

**ISSUES IN RECURSIVE DYNAMIC CGE MODELING :
INVESTMENT BY DESTINATION, SAVINGS, AND PUBLIC DEBT
A Survey**

by
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Université du Québec, INRS-Urbanisation, Culture et Société
CIRPÉE, Université Laval

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CONTENTS

Contents	i
Part one : Investment by DESTINATION	1
Introduction	3
1. Investment demand theory	3
1.1 Nickell's continuous-time theoretical model	4
1.1 Nickell's basic model in discrete time	6
1.2 A model with adjustment costs	16
2. Applied models of investment demand	26
2.1 Bourguignon, Branson and de Melo, J. (1989)	26
2.2 Jung and Thorbecke (2001)	29
2.3 Agénor (2003)	31
2.4 Fargeix and Sadoulet (1994)	31
2.5 Collange (1993)	32
2.6 Investment demand in CGE models	33
3. Non-demand models of the distribution of investment by destination	33
3.1 A sketch of an investment supply model	34
3.2 Beghin, Dessus, Roland and Mensbrugghe (1996)	38
3.3 The <i>MIRAGE</i> model of Bchir, Decreux, Guérin and Jean (2002)	40
3.4 Abbink, Braber and Cohen (1995)	48
3.5 Thurlow (2003), and Dervis, de Melo and Robinson (1982)	49
3.6 Dumont and Mesplé-Somps (2000)	52
4. Synthesis of the theory and surveyed applications	52
4.1 Overview of available choices	52
4.2 Links with the issues of savings and debt	56
4.3 Conclusion	56
Part one references	59
Appendix A1 : A theoretical model with first-degree homogenous adjustment costs	63
A1.1 First-order optimum conditions	63
A1.2 User cost of capital with adjustment costs	65
A1.3 Tobin's q in the first-order conditions	66
A1.4 the intertemporal equilibrium of capital	70

A1.5 Investment demand with stationary expectations	73
Annexe A2 : Mathematical developments	77
A2.1 Basic discrete-time model : first order optimum conditions	77
A2.2 Basic discrete-time model : Tobin's q	79
A2.3 Basic discrete-time model : the intertemporal equilibrium of capital	81
A2.4 Basic discrete-time model : the user cost of capital with stationary expectations	82
A2.5 Model with adjustment costs : first-order optimum conditions	83
A2.6 Model with adjustment costs : Tobin's q	85
A2.7 Model with adjustment costs : the intertemporal equilibrium of capital	87
Part two : Savings	91
Introduction	93
1. The Extended Linear Expenditure System (ELES)	95
2. The price of future consumption and the rate of return on savings	97
3. Savings made sensitive to the rate of return	98
4. The SELES model : savings sensitive to the rate of return, with endogenous labor supply	102
5. Summary and calibration considerations	104
Part two references	107
Appendix B : Derivation of the demand functions of the SELES model	111
Part three : Public debt	115
1. Issues related to debt in a CGE	117
1.1 Objective	117
1.2 Basic requirements	117
2. Literature	118
2.1 Thissen's (1999) survey	118
2.2 The « Maquette » model of Bourguignon, Branson and de Melo (1989)	119
2.3 The Rosensweig and Taylor (1990) model	120
2.4 Collange's (1993) Ivory Coast model	123
2.5 Decaluwé-Souissi's (1994) and Souissi's (1994) model	124
2.6 The Lemelin (2005, 2007) model	126
Part three references	127
Compendium of mathematical expressions	129

PART ONE :
INVESTMENT BY DESTINATION

Introduction

In a recursive dynamic model, each period's static equilibrium determines the amount of savings and, therefore, the total amount of investment spending. It remains to be known how investment expenditures are distributed between industries. For the stock of capital available in every period t in every industry i is determined by the intertemporal constraint

$$K_{i,t+1} = I_{i,t} + (1 - \delta_i)K_{i,t} \quad [001]$$

In this first part of the document, we concentrate precisely on the variable $I_{i,t}$ in the preceding equation. This part of the document comprises three chapters, plus an introduction and a summary-conclusion.

The first chapter reviews the economic theory of the firm's investment demand, as it is formulated by Nickell (1978), and then translates it to a discrete time framework to make it applicable to a dynamic recursive model. The second chapter surveys several applied models of investment demand implemented in CGE's. This survey also attempts to evaluate these models in light of the theory, and to describe how savings and the sum of investments are balanced. The third chapter examines another family of models, namely models of the distribution of investment among industries, some of which are akin to models of the supply of capital. The fourth, final chapter proposes a synthesis of the theory and of the surveyed models.

1. Investment demand theory

There are few CGE models where the mechanism for distributing investment among industries has explicit theoretical foundations. Bourguignon *et al.* (1989) are an exception. They refer to Nickell's (1978) model in the following terms: « Such a functional form is consistent with formulations of investment demand in which there are costs of adjustment and investment decisions are irreversible ». So it would seem appropriate, both to understand the theoretical foundations of the Bourguignon *et al.* (1989) specification, and to establish a theoretical framework, to go back to basics, and examine Nickell's (1978) model. All the more so, given that Nickell's presentation is very systematic and well worth studying.

Nickell's model is formulated in continuous time. After presenting the basic model in continuous time, we will transpose it to discrete time, so that the results are usable in the context of a recursive dynamic model. We then develop a version of the model with adjustment costs which, under certain conditions, leads to the theoretical form invoked by Bourguignon *et al.* (1989).

1.1 NICKELL'S CONTINUOUS-TIME THEORETICAL MODEL

Nickell's basic model rests upon the following hypotheses :

1. The market for capital is perfect (which implies, among others, the possibility of borrowing or lending without limit).
2. The future is known with certainty.
3. The firm produces a single good by combining labor and capital according to a doubly-differentiable production function, with strictly decreasing returns to scale everywhere.
4. Capital is not distinguishable by its production date (no vintage effect).
5. The firm is a price taker on all markets; in particular, it can sell or buy capital at any time without restriction, and capital becomes productive instantaneously at the moment of purchase.
6. There is no cost to selling, buying or installing new capital.
7. There are no taxes; in particular, there are no taxes on capital, neither are there investment subsidies.

The firm's instantaneous cash flow is the excess of gross revenue from the sale of output over salaries paid and investment expenditures :

$$p_t F(K_t, L_t) - w_t L_t - q_t I_t \quad [002]$$

where

L_t is the volume of labor employed at time t

K_t is the stock of capital in place at time t

I_t is the volume of investment at time t

p_t is output price at time t

w_t is the wage rate at time t

q_t is the replacement cost of capital, or, equivalently, the price of the investment good, at time t

$F(K_t, L_t)$ is the production function

The dynamic constraint linking capital to investment is

$$\dot{K}_t = I_t - \delta K_t \quad [003]$$

where the dot over a variable indicates its time derivative, and δ is the instantaneous rate of depreciation.

The firm maximizes the present value of its cash flow. Assuming discount rate r to be constant, the maximization problem is :

$$MAX V = \int_0^{\infty} e^{-rt} [p_t F(K_t, L_t) - w_t L_t - q_t I_t] dt \quad [004]$$

$$\text{s.t. } \dot{K}_t = I_t - \delta K_t \quad [003]$$

$$\text{and } K_0 = \bar{K}_0 \quad [005]$$

The solution leads to first-order conditions

$$p_t \frac{\partial F}{\partial L_t} = w_t \quad [006]$$

$$p_t \frac{\partial F}{\partial K_t} = q_t \left[r + \delta - \frac{\dot{q}_t}{q_t} \right] \quad [007]$$

$$I_t = \dot{K}_t + \delta K_t \quad [003]$$

$$K_0 = \bar{K}_0 \quad [005]$$

The right-hand side of [007] is the user-cost of capital.

Defining the instantaneous inflation rate

$$\pi_t = \frac{\dot{q}_t}{q_t} \quad [008]$$

we find the usual expression for the user-cost of capital, noted as u_t :

$$u_t = q_t (r + \delta - \pi_t) \quad [009]$$

As Nickell (1978, p. 11) points out, the firm's dynamic problem reduces to an essentially static solution : at every moment, the optimal conditions depend only on current values of variables and their rates of change. As a matter of fact, it is possible to define an instantaneous maximization problem whose solution is equivalent to that of problem [004] subject to [003] and [005] :

$$MAX [p_t F(K_t, L_t) - w_t L_t - u_t K_t] \quad [010]$$

$$\text{s.c. } K_t = \bar{K}_0 + \int_0^t \dot{K}_\tau d\tau = \bar{K}_0 + \int_0^t (I_\tau - \delta K_\tau) d\tau \quad [011]$$

First-order conditions are

$$p_t \frac{\partial F}{\partial L_t} = w_t \quad [006]$$

$$p_t \frac{\partial F}{\partial K_t} = u_t \quad [012]$$

$$I_t = \dot{K}_t + \delta K_t \quad [003]$$

$$K_0 = \bar{K}_0 \quad [005]$$

It is easily verified that, given [008] and [009], conditions [006], [012], [003] and [005] are strictly equivalent to [006], [007], [003] and [005] .

1.1 NICKELL'S BASIC MODEL IN DISCRETE TIME

1.1.1 First-order optimal conditions

Period t 's cash-flow is the excess of gross revenue from the sale of output, over salaries paid and investment expenditures :

$$p_t F(K_t, L_t) - w_t L_t - q_t I_t \quad [002]$$

where

L_t is the volume of labor employed in period t

K_t is capital stock in period t

I_t is the volume of investment in period t

p_t is output price in period t

w_t is period t wage rate

q_t is the replacement cost of capital or, equivalently, the price of the investment good, in period t

$F(K_t, L_t)$ is the production function

The intertemporal constraint linking capital and investment is given by

$$\Delta_t K = K_{t+1} - K_t = I_t - \delta K_t \quad [013]$$

or, equivalently,

$$I_t = K_{t+1} - K_t + \delta K_t = K_{t+1} - (1 - \delta) K_t \quad [014]$$

where δ is the per period rate of depreciation.

That formulation means that capital installed during period t can be used only in the following period. So this way of transposing the model to discrete time amounts to introducing a capital installation lag. In continuous time, as mentioned earlier (Nickell, 1978, p. 11), the firm's dynamic problem reduces to an essentially static solution, where each moment's optimal conditions depend only on current values of the variables. But in discrete time, each period's optimal conditions depend on current *and past* values of the variables. And as past values constrain the present, so current decisions will constrain the future : in every period, the firm must therefore take the future into account and make the decisions which will make it possible to verify the optimum conditions in the future¹ : the capital accumulation constraint is truly *intertemporal*.

Note that, in [014], depreciation is applied between periods, so that the amount of capital available at the beginning of the period remains so throughout the period. Nickell's hypothesis 5 needs to be restated as :

5. The firm is a price taker on all markets; in particular, it can buy or sell any quantity of capital at the end of every period t , and that capital becomes or ceases to be available at the beginning of the following period.

The firm maximizes the present value of its cash-flow. Assuming discount rate r to be constant, the maximization problem is :

$$MAX V = \sum_{t=0}^{\infty} \frac{1}{(1+r)^t} [p_t F(K_t, L_t) - w_t L_t - q_t I_t] \quad [015]$$

$$\text{s.c. } I_t = K_{t+1} - K_t + \delta K_t = K_{t+1} - (1-\delta)K_t \quad [014]$$

$$\text{and } K_0 = \bar{K}_0 \quad [005]$$

The solution leads to first-order conditions (see details in Annex A2.1)

$$p_t \frac{\partial F}{\partial L_t} = w \quad [016]$$

$$p_t \frac{\partial F}{\partial K_t} = (1+r)q_{t-1} - q_t(1-\delta) \quad [017]$$

$$I_t = K_{t+1} - K_t + \delta K_t = K_{t+1} - (1-\delta)K_t \quad [014]$$

$$K_0 = \bar{K}_0 \quad [005]$$

¹ Unless it is assumed, as in MIRAGE (Bchir et al., 2002), that investment is instantaneously productive.

Condition [017] can be rewritten as

$$p_t \frac{\partial F}{\partial K_t} = r q_{t-1} + \delta q_t - (q_t - q_{t-1}) \quad [018]$$

Note the presence of a lagged variable in [017] and [018]. That comes from the new capital installation delay in discrete-time intertemporal constraint [014].

1.1.2 The discrete-time user-cost of capital

Define the retrospective rate of inflation²

$$\pi_t = \frac{(q_t - q_{t-1})}{q_{t-1}} \quad [019]$$

so that

$$(q_t - q_{t-1}) = \frac{(q_t - q_{t-1})}{q_{t-1}} q_{t-1} = \pi_t q_{t-1} \quad [020]$$

and condition [018] becomes

$$p_t \frac{\partial F}{\partial K_t} = r q_{t-1} + \delta q_t - \pi_t q_{t-1} = (r - \pi_t) q_{t-1} + \delta q_t \quad [021]$$

The right-hand side of [021]

$$u_t = (r - \pi_t) q_{t-1} + \delta q_t \quad [022]$$

is the user-cost of capital to which the marginal value product must be equal at optimum. But user-cost [022] takes on a different form. compared to its usual continuous-time form $q_t(r + \delta - \pi_t)$.

One difference between the right-hand side of [022] and the continuous-time user-cost of capital is that, in discrete time, the instantaneous inflation rate is replaced by a periodical inflation rate. The second difference is the presence of lagged variable q_{t-1} and, more importantly, of the product $r q_{t-1}$: just like in [017] and [018], that comes from the new capital installation delay in discrete-time intertemporal constraint [014].

² We could just as well define a prospective rate of inflation, without substantially changing results.

1.1.3 Tobin's « q » in the first-order conditions

N.B. : In the following lines, the notation used is close to that of Tobin (1969) and Tobin and Brainard (1977), but translated to discrete time.

Tobin (1969) defines the « q » ratio as

$$q = \frac{\text{Market value of the firm}}{\text{Replacement cost of capital}}$$

The firm's market value, or stock market capitalization, corresponds to the present value of the stream of future income that shareholders expect. In a simplified universe, without inflation, or taxes, or depreciation, a quantity ΔK of capital, with an acquisition cost of $p \Delta K$, yields a stream of expected incomes $\{E(t)\}$.

The marginal efficiency of capital ρ is defined implicitly³. The value of ρ is the one that solves

$$p \Delta K = \sum_{t=1}^{\infty} \left(\frac{1}{1+\rho} \right)^t E(t) \quad [023]$$

In the particular instance where $E(t)$ is constant, equal to \bar{E} , it follows that

$$p \Delta K = \sum_{t=1}^{\infty} \left(\frac{1}{1+\rho} \right)^t E(t) = \sum_{t=1}^{\infty} \left(\frac{1}{1+\rho} \right)^t \bar{E} = \frac{1}{\rho} \bar{E} \quad [024]$$

Thus, marginal efficiency of capital ρ can be defined as the rate which must be applied to the cost of investment $p \Delta K$ for a perpetual rent in the amount of $\bar{E} = \rho p \Delta K$ to have the same present value as the stream of expected incomes $\{E(t)\}$.

On the other hand, if MV is the stock market evaluation of that investment, then the rate of return implicitly demanded by the market for shares is

$$MV = \sum_{t=1}^{\infty} \left(\frac{1}{1+r_K} \right)^t E(t) \quad [025]$$

where it is assumed that the sequence of incomes $\{E(t)\}$ expected by the market is the same as expected by the promotor.

In the particular case of constant $E(t)$, equal to \bar{E} , [025] becomes

³ Tobin (1969) and Tobin et Brainard (1977) use the symbol R . We replaced it with ρ to avoid confusion with R_t , which will be used later on to designate the marginal value product of capital.

$$MV = \sum_{t=1}^{\infty} \left(\frac{1}{1+r_K} \right)^t E(t) = \sum_{t=1}^{\infty} \left(\frac{1}{1+r_K} \right)^t \bar{E} = \frac{1}{r_K} \bar{E} \quad [026]$$

Rate of return r_K implicitly demanded by the market is therefore the rate that must be applied to the market evaluation of investment MV for a perpetual rent of $\bar{E} = r_K MV$ to have the same present value as sequence $\{E(t)\}$ of expected incomes.

Tobin's « q » is the ratio of the market value of the investment on its replacement cost :

$$q = \frac{MV}{\rho \Delta K} \quad [027]$$

It is profitable to invest when the value of q is greater than 1. More rigorously, the optimal volume of investment is the one at which the marginal increase in the market value of investment is equal to its marginal replacement cost⁴.

In the particular case of constant $E(t)$, equal to \bar{E} , [027] becomes

$$q = \frac{MV}{\rho \Delta K} = \frac{\left(\frac{\bar{E}}{r_K} \right)}{\left(\frac{\bar{E}}{\rho} \right)} = \frac{\left(\frac{1}{r_K} \right)}{\left(\frac{1}{\rho} \right)} = \frac{\rho}{r_K} \quad [028]$$

In the general case, when $E(t)$ is not necessarily constant, MV is nonetheless a monotonically decreasing function of the rate of return demanded by the market, r_K . Now, for a given value of $\rho \Delta K$, and a given sequence $\{E(t)\}$ of expected incomes, the marginal efficiency of capital is fixed. It follows that the value of q is a monotonically decreasing function of r_K , just like MV , and,

by the same token, a monotonically decreasing function of the ratio $\frac{\rho}{r_K}$.

Moreover, in view of equations [023] and [025], it is clear that Tobin's q is equal to 1 when $r_K = \rho$.

N.B. : We now return to the notation used in the present paper.

⁴ The ratio of the marginal increase of the market value over the marginal cost of investment is what is referred to in the literature as the « marginal q » **provided the marginal cost of investment is constant**. If, however, adjustment costs increase with the volume of investment, it is incorrect to identify that ratio as the « marginal q », because, in that case, it is the ratio of two derivatives, not the derivative of a ratio (the q ratio).

Where is Tobin's q in the model presented above? Develop⁵ :

$$p_t \frac{\partial F}{\partial K_t} = (1+r)q_{t-1} - (1-\delta)q_t \quad [017]$$

and it follows (see details in Appendix A2.2)

$$q_t K_{t+1} = \frac{1}{(1+r)} \left[(1-\delta)q_{t+1} K_{t+1} + p_{t+1} \frac{\partial F}{\partial K_{t+1}} K_{t+1} \right] \quad [029]$$

Now, the accumulation rule can be written as

$$(1-\delta)K_t = K_{t+1} - I_t \quad [030]$$

Substituting into [029] obtains

$$q_t K_{t+1} = \frac{1}{(1+r)} \left(p_{t+1} \frac{\partial F}{\partial K_{t+1}} K_{t+1} - q_{t+1} I_{t+1} + q_{t+1} K_{t+2} \right) \quad [031]$$

where the term $q_{t+1} K_{t+2}$ can be replaced with its expression following the same equation [031].

Successive substitutions lead to

$$q_t K_{t+1} = \sum_{s=1}^{\infty} \frac{1}{(1+r)^s} \left[p_{t+s} \frac{\partial F}{\partial K_{t+s}} K_{t+s} - q_{t+s} I_{t+s} \right] + \lim_{s \rightarrow \infty} \left(\frac{1}{(1+r)^s} q_{t+s} K_{t+s+1} \right) \quad [032]$$

where the last term is zero under the transversality condition

$$\lim_{s \rightarrow \infty} \frac{1}{(1+r)^s} q_{t+s} K_{t+s+1} = 0 \quad [033]$$

Transversality condition⁶

Begin with the following T -period finite-horizon problem

$$MAX V = \sum_{t=0}^T \frac{1}{(1+r)^t} \left[p_t F(K_t, L_t) - w_t L_t - q_t (K_{t+1} - (1-\delta)K_t) \right] \quad [034]$$

$$\text{s.c. } K_0 = \bar{K}_0 \quad [005]$$

The objective condition can be written as

⁵ The following development is parallel to Hayashi (1982), as reproduced in Nabil Annabi, *Les MÉGC avec anticipations rationnelles : introduction*, présentation diapo, mars 2003; see slides No. 38 and following.

⁶ Source : Murat Yildizoglu, Notes de cours de croissance économique, Université de Bordeaux 4, mars 1999
<http://yildizoglu.u-bordeaux4.fr/croisemfweb/croisemfweb.html>
<http://yildizoglu.u-bordeaux4.fr/croisemfweb/node21.html>

$$MAX V = \sum_{t=0}^T \frac{1}{(1+r)^t} [p_t F(K_t, L_t) - w_t L_t] - \sum_{t=0}^T \frac{1}{(1+r)^t} q_t K_{t+1} + \sum_{t=0}^T \frac{1}{(1+r)^t} q_t (1-\delta) K_t \quad [035]$$

Since production in $T+1$ is ruled out, and that, by hypothesis, capital can be sold at the end of T , nothing forces remaining capital K_{T+1} to be strictly positive. It follows that a program cannot be optimal if the last term of the second sum is positive, that is, if

$$\frac{1}{(1+r)^T} q_T K_{T+1} > 0 \quad [036]$$

Either remaining capital is zero, either its value is null. Whence, the *transversality* condition, that is, the horizon-crossing (traversing) condition :

$$\frac{1}{(1+r)^T} q_T K_{T+1} = 0 \quad [037]$$

That condition remains true, no matter how far horizon T . When the horizon is indefinitely far, any optimal program must therefore respect

$$\lim_{T \rightarrow \infty} \frac{1}{(1+r)^T} q_T K_{T+1} = 0 \quad [038]$$

Of course, that implies, for $s = T - t$,

$$\lim_{s \rightarrow \infty} \frac{1}{(1+r)^{t+s}} q_{t+s} K_{t+s+1} = \frac{1}{(1+r)^t} \lim_{s \rightarrow \infty} \frac{1}{(1+r)^s} q_{t+s} K_{t+s+1} = 0 \quad [033]$$

So

$$q_t K_{t+1} = \sum_{s=1}^{\infty} \frac{1}{(1+r)^s} \left[p_{t+s} \frac{\partial F}{\partial K_{t+s}} K_{t+s} - q_{t+s} l_{t+s} \right] \quad [039]$$

We now depart from Nickell's (1978) hypotheses, and, instead of strictly decreasing returns to scale, we suppose constant returns. Production function $F(K_t, L_t)$ is then first-degree homogenous, which implies Euler's condition

$$F(K_t, L_t) = \frac{\partial F}{\partial K_t} K_t + \frac{\partial F}{\partial L_t} L_t \quad [040]$$

Or, in view of first-order condition [016],

$$p_t \frac{\partial F}{\partial K_t} K_t = p_t F(K_t, L_t) - w_t L_t \quad [041]$$

Equation [039] can thus be rewritten as

$$q_t K_{t+1} = \sum_{s=1}^{\infty} \frac{1}{(1+r)^s} [p_{t+s} F(K_{t+s}, L_{t+s}) - w_{t+s} L_{t+s} - q_{t+s} I_{t+s}] \quad [042]$$

or

$$\frac{\sum_{s=1}^{\infty} \frac{1}{(1+r)^s} [p_{t+s} F(K_{t+s}, L_{t+s}) - w_{t+s} L_{t+s} - q_{t+s} I_{t+s}]}{q_t K_{t+1}} = 1 \quad [043]$$

The numerator of [043] is the present value in period t of the enterprise's cash flow from $t+1$ onwards; the discount rate is the market rate. Therefore, that present value corresponds to the stock market valuation that is the numerator of Tobin's q . Take note of the one-period time lag : capital available in period $t+1$ must have been invested in period t (or re-invested, that is, not disinvested); thus, the cash flows to be taken into account are from period $t+1$. The denominator of the left-hand side of [043] is the replacement cost in period t of capital to be used from period $t+1$ (note : *replacement* cost, not user cost). So the left-hand side ratio of [043] is indeed Tobin's q : investment made in t is optimal when that ratio is 1.

1.1.4 Intertemporal equilibrium of capital

Making successive substitutions for $t+1$, $t+2$, etc., in condition [282], results in (see détails in Appendix A2.3) :

$$q_t = \frac{1}{(1+r)} \left\{ \begin{array}{l} \frac{(1-\delta)^3}{(1+r)^2} q_{t+3} + \frac{(1-\delta)^2}{(1+r)^2} p_{t+3} \frac{\partial F}{\partial K_{t+3}} \\ + \frac{(1-\delta)}{(1+r)} p_{t+2} \frac{\partial F}{\partial K_{t+2}} \\ + p_{t+1} \frac{\partial F}{\partial K_{t+1}} \end{array} \right\} \quad [044]$$

or, after developing :

$$q_t = \left(\frac{1-\delta}{1+r} \right)^\theta q_\theta + \frac{1}{(1+r)} \sum_{s=1}^{\theta} \left(\frac{1-\delta}{1+r} \right)^{s-1} p_{t+s} \frac{\partial F}{\partial K_{t+s}} \quad [045]$$

The first of the two right-hand side terms is the present value of the proceeds from selling, in period θ , one unit of capital acquired in period t , after going through $\theta-t$ periods of depreciation. The second term is the present value of the flow of marginal products of capital from period $t+1$ until period θ .

Given that the horizon of maximization problem [015] is indefinitely far, we can make θ tend to infinity, and get

$$q_t = \lim_{\theta \rightarrow \infty} \left(\frac{1-\delta}{1+r} \right)^\theta q_\theta + \frac{1}{(1+r)} \sum_{s=1}^{\infty} \left(\frac{1-\delta}{1+r} \right)^{s-1} p_{t+s} \frac{\partial F}{\partial K_{t+s}} \quad [046]$$

In that equation, the first right-hand side term must be null. First, since prices q_t and discount rate r are positive, it cannot be negative, unless depreciation rate δ is greater than 1, which would be absurd. Next, for that term to be positive, it would be necessary that, in the long run, price of capital q_t grow at a rate greater than

$$\frac{1+r}{1-\delta} - 1 = \frac{r+\delta}{1-\delta} \quad [047]$$

Intuitively, such a rising tendency for price q_t would be an incentive for the firm to acquire capital with the aim of making a speculative profit upon resale. More rigorously, if

$$\lim_{\theta \rightarrow \infty} \left(\frac{1-\delta}{1+r} \right)^\theta q_\theta > 0 \quad [048]$$

equation [046] implies

$$q_t > \frac{1}{(1+r)} \sum_{s=1}^{\infty} \left(\frac{1-\delta}{1+r} \right)^{s-1} p_{t+s} \frac{\partial F}{\partial K_{t+s}} \quad [049]$$

That means that the firm is ready to pay for a marginal unit of capital more than the present value of the marginal product it can generate if it is kept forever. That is exactly the definition of speculation given by Harrison and Kreps (1975, quoted by Tirole, 1982), following Kaldor and Keynes : « investors exhibit speculative behavior if the right to resell [an] asset makes them willing to pay more for it than they would pay if obliged to hold it forever ».

To exclude that possibility, we impose the condition of no speculative bubbles

$$\lim_{t \rightarrow \infty} \left(\frac{1-\delta}{1+r} \right)^t q_t = 0 \quad [050]$$

That condition turns [046] into

$$q_t = \frac{1}{(1+r)} \sum_{s=1}^{\infty} \left(\frac{1-\delta}{1+r} \right)^{s-1} p_{t+s} \frac{\partial F}{\partial K_{t+s}} \quad [051]$$

Equilibrium condition [051], like condition [017] of which it is a consequence, underlines the intertemporal nature of the producer's problem. For, under the competition assumption that the

producer is a price-taker, he/she has influence neither on capital replacement price q_t , nor on future product prices p_{t+s} . The only way to fulfill [051] is, depending on anticipated prices p_{t+s} , to adjust the marginal product of capital in future periods, $\frac{\partial F}{\partial K_{t+s}}$; in order to do that, he/she must manage the evolution of the stock of capital, that is, make in every period the necessary investments.

1.1.5. User cost of capital with stationary expectations

We shall demonstrate that, under the hypothesis of stationary expectations, the user cost of capital returns to its usual form in the absence of inflation.

To simplify notation, let us denote the value of the marginal product of capital as

$$R_t = p_t \frac{\partial F}{\partial K_t} \quad [052]$$

Equation [051] becomes

$$q_t = \frac{1}{(1+r)} \sum_{s=0}^{\infty} \left(\frac{1-\delta}{1+r} \right)^s R_{t+s+1} \quad [053]$$

If, in that expression, we replace R_{t+s+1} with \tilde{R}_{t+s} , its anticipated value at time t , and if we assume that value to be constant (stationary expectations), then

$$\tilde{R}_{t+s} = R_t, \forall s \geq 0 \quad [054]$$

and condition [053] becomes :

$$R_t = (r + \delta) q_t \quad [055]$$

(details are found in Appendix A2.4)

The right-hand side of [055] is the usual form of the user cost of capital in the absence of replacement cost inflation :

$$\bar{u}_t = (r + \delta) q_t \quad [056]$$

Indeed, under those circumstances,

$$\pi_t = \frac{(q_t - q_{t-1})}{q_{t-1}} = 0, \text{ which implies } q_t = q_{t-1} \quad [057]$$

In short, with stationary expectations [054], user cost of capital [022] returns to its usual form [056] and equilibrium condition [018] is

$$R_t = (r + \delta)q_t = \bar{u}_t \quad [058]$$

1.2 A MODEL WITH ADJUSTMENT COSTS

Nickell (1978, Chapter 3) develops a theoretical model with adjustment costs⁷, with the following hypothesis :

8. There are adjustment costs associated with variations in the stock of capital. These costs are a function of gross investment, they grow with the absolute value of the *rate* of investment or disinvestment, and, moreover, grow at an increasing rate. They are null only when gross investment is zero.

Formally, that implies an adjustment cost function $C(I_t)$ with the following properties (Nickell, 1978, p. 27) :

$$C'(I_t) > \text{or} < 0 \Leftrightarrow I_t > \text{or} < 0 \quad [059]$$

$$C(0) = 0 \quad [060]$$

$$C''(I_t) > 0 \quad [061]$$

Among the functional forms which possess these properties, is⁸ :

$$C(I_t) = q_t \frac{\gamma}{2} I_t^2 \quad [062]$$

It is a model where adjustment costs are independent of the stock of capital. In Appendix 1, a model is presented where adjustment costs are a function of the volume of investments, and inversely proportional to the stock of capital.

⁷ Épaulard (1993) uses the expression « installation costs ».

⁸ Note that cost function [062] is sometimes stated as

$$C(I_t) = \frac{\gamma}{2} I_t^2$$

in which case [202] becomes

$$MAX V = \sum_{t=0}^{\infty} \frac{1}{(1+r)^t} [p_t F(K_t, L_t) - w_t L_t - q_t I_t - q_t C(I_t)]$$

1.2.1 First-order optimum conditions

As before, the firm maximizes the present value of its cash flow. Assuming a constant discount rate, the maximization problem is :

$$MAX V = \sum_{t=0}^{\infty} \frac{1}{(1+r)^t} [p_t F(K_t, L_t) - w_t L_t - q_t I_t - C(I_t)] \quad [063]$$

$$\text{s.t. } I_t = K_{t+1} - K_t + \delta K_t = K_{t+1} - (1-\delta)K_t \quad [014]$$

$$\text{and } K_0 = \bar{K}_0 \quad [005]$$

The solution leads to the following first-order conditions (see details in Appendix A2.5) :

$$p_t \frac{\partial F}{\partial L_t} = w \quad [016]$$

$$p_t \frac{\partial F}{\partial K_t} = (1+r)q_{t-1}(1+\gamma I_{t-1}) - q_t(1+\gamma I_t)(1-\delta) \quad [064]$$

$$I_t = K_{t+1} - (1-\delta)K_t \quad [014]$$

$$K_0 = \bar{K}_0 \quad [005]$$

Let

$$Q_t = \frac{\partial}{\partial I_t} \left[q_t I_t \left(1 + \frac{\gamma}{2} I_t \right) \right] = q_t (1 + \gamma I_t) \quad [065]$$

It is the marginal cost, or implicit price, of investment. Substituting [065] into [064], we write

$$p_t \frac{\partial F}{\partial K_t} = (1+r)Q_{t-1} - (1-\delta)Q_t \quad [066]$$

1.2.2 User cost of capital with adjustment costs

Define the retrospective rate of increase in the marginal cost of capital

$$\Pi_t = \frac{(Q_t - Q_{t-1})}{Q_{t-1}} \quad [067]$$

so that

$$(Q_t - Q_{t-1}) = \frac{(Q_t - Q_{t-1})}{Q_{t-1}} Q_{t-1} = \Pi_t Q_{t-1} \quad [068]$$

And [066] can be written as

$$p_t \frac{\partial F}{\partial K_t} = r Q_{t-1} + \delta Q_t - (Q_t - Q_{t-1}) \quad [069]$$

$$p_t \frac{\partial F}{\partial K_t} = (r - \Pi_t) Q_{t-1} + \delta Q_t \quad [070]$$

The right-hand side membre of that equation is the value of the marginal product of capital in period t . The left-hand side member, on the other hand, is the user cost of capital when there are adjustment costs :

$$U_t = (r - \Pi_t) Q_{t-1} + \delta Q_t \quad [071]$$

So we can write [066] as

$$p_t \frac{\partial F}{\partial K_t} = U_t \quad [072]$$

Equation [072] is the equivalent of [021], modified in order to take into account the adjustment costs.

1.2.3 Tobin's « q » in the first-order conditions

Is it possible to find Tobin's q in the model just developed?

Let the value of the marginal productivity of capital be

$$R_t = p_t \frac{\partial F}{\partial K_t} \quad [052]$$

and develop [066]; there results (see details in Appendix A2.6)⁹ :

$$Q_t K_{t+1} = \frac{1}{(1+r)} [(1-\delta) Q_{t+1} K_{t+1} + R_{t+1} K_{t+1}] \quad [073]$$

Replace $(1-\delta) K_{t+1}$ with its equivalent according to [030] (which is a reformulation of accumulation constraint [014]), and we find

$$Q_t K_{t+1} = \frac{1}{(1+r)} (R_{t+1} K_{t+1} - Q_{t+1} I_{t+1} + Q_{t+1} K_{t+2}) \quad [074]$$

where the term $Q_{t+1} K_{t+2}$ can be replaced by its expression according to that very same equation. Follows

⁹ The following development is parallel to Hayashi (1982), as reproduced in Nabil Annabi, *Les MÉGC avec anticipations rationnelles : introduction*, présentation diapo, mars 2003; see slides No. 38 and following.

$$Q_t K_{t+1} = \frac{1}{(1+r)} \left[\frac{R_{t+1} K_{t+1} - Q_{t+1} I_{t+1}}{(1+r)} + \frac{1}{(1+r)} (R_{t+2} K_{t+2} - Q_{t+2} I_{t+2} + Q_{t+2} K_{t+3}) \right] \quad [075]$$

Making successive substitutions of $Q_{t+s} K_{t+s+1}$ leads to

$$Q_t K_{t+1} = \sum_{s=1}^{\infty} \frac{1}{(1+r)^s} [R_{t+s} K_{t+s} - Q_{t+s} I_{t+s}] + \lim_{s \rightarrow \infty} \left(\frac{1}{(1+r)^s} Q_{t+s} K_{t+s+1} \right) \quad [076]$$

where the last term is null by the transversality condition

$$\lim_{s \rightarrow \infty} \left(\frac{1}{(1+r)^s} Q_{t+s} K_{t+s+1} \right) = 0 \quad [077]$$

Therefore

$$Q_t K_{t+1} = \sum_{s=1}^{\infty} \frac{1}{(1+r)^s} [R_{t+s} K_{t+s} - Q_{t+s} I_{t+s}] \quad [078]$$

As before, we depart from Nickell's (1978) assumptions, and, instead of strictly decreasing returns to scale, we assume constant returns. Production function $F(K_t, L_t)$ is then first-degree homogenous, which implies Euler's condition [040]. Given first-order condition [016], we have

$$R_t K_t = p_t F(K_t, L_t) - w_t L_t \quad [079]$$

Replace $R_{t+s} K_{t+s}$ in [078] :

$$Q_t K_{t+1} = \sum_{s=1}^{\infty} \frac{1}{(1+r)^s} [p_{t+s} F(K_{t+s}, L_{t+s}) - w_{t+s} L_{t+s} - Q_{t+s} I_{t+s}] \quad [080]$$

Then substitute definition [065] for Q_t . After some rearranging (details in Appendix A2.6), we get to

$$\frac{\sum_{s=1}^{\infty} \frac{1}{(1+r)^s} \left[p_{t+s} F(K_{t+s}, L_{t+s}) - w_{t+s} L_{t+s} - q_{t+s} I_{t+s} \left(1 + \frac{\gamma}{2} I_{t+s} \right) \right]}{Q_t K_{t+1}} = 1 + \frac{\sum_{s=1}^{\infty} \frac{1}{(1+r)^s} \left[q_{t+s} I_{t+s} \left(\frac{\gamma}{2} I_{t+s} \right) \right]}{Q_t K_{t+1}} \quad [081]$$

The numerator of the left-hand side member of [081] is the present value in period t of the firm's cash flows from period $t+1$ onwards; the discount rate is the market rate. Therefore, that present value corresponds to the stock market valuation that is the numerator of Tobin's q . Take note of the one-period time lag : capital available in period $t+1$ must have been invested in period t (or re-invested, that is, not disinvested); thus, the cash flows to be taken into account are from

period $t+1$. The denominator of the left-hand side of [081] is the replacement cost in period t of capital to be used from period $t+1$ (note : *replacement* cost, not user cost). So the left-hand side ratio of [081] is analog to Tobin's q . But, contrary to Tobin's q , the denominator of that ratio is not a constant price, but a marginal cost that takes into account adjustment costs. Moreover, contrary to Tobin's result, the level of investment in t is not optimal when it is equal to 1, but rather when it attains a certain value greater to 1, as shown by the right-hand side of [081]. In comparison with Tobin's rule, this model leads to less investment. That is due to the fact that the adjustment cost is independent of the stock of capital, so that, since the adjustment cost function is not first-degree homogenous, the conditions required by Hayashi (1982) are not fulfilled. On the other hand, it can be shown that, if the adjustment cost function were of the form

$$C(I_t, K_t) = q_t \frac{\gamma}{2} \frac{I_t^2}{K_t} \quad [082]$$

then the conditions of Hayashi (1982) are fulfilled, and a similar development to what precedes leads to equilibrium condition

$$\frac{\sum_{s=1}^{\infty} \frac{1}{(1+r)^s} \left[p_{t+s} F(K_{t+s}, L_{t+s}) - w_{t+s} L_{t+s} - q_{t+s} I_{t+s} \left(1 + \frac{\gamma}{2} \frac{I_{t+s}}{K_{t+s}} \right) \right]}{Q_t K_{t+1}} = 1 \quad [083]$$

where the left-hand side member is identical to [081] (details in Appendix A1).

1.2.4 The intertemporal equilibrium of capital

Condition [066] is equivalent to

$$Q_t = \frac{(1+r)}{(1-\delta)} Q_{t-1} - \frac{1}{(1-\delta)} p_t \frac{\partial F}{\partial K_t} \quad [084]$$

which can be rewritten as

$$Q_t = \frac{1}{(1+r)} \left\{ (1-\delta) Q_{t+1} + p_{t+1} \frac{\partial F}{\partial K_{t+1}} \right\} \quad [085]$$

Successive substitutions for $t+1$, $t+2$, etc., lead to (details in Appendix A2.7) :

$$Q_t = \frac{1}{(1+r)} \left\{ \begin{aligned} & \frac{(1-\delta)^3}{(1+r)^2} Q_{t+3} + \frac{(1-\delta)^2}{(1+r)^2} p_{t+3} \frac{\partial F}{\partial K_{t+3}} \\ & + \frac{(1-\delta)}{(1+r)} p_{t+2} \frac{\partial F}{\partial K_{t+2}} \\ & + p_{t+1} \frac{\partial F}{\partial K_{t+1}} \end{aligned} \right\} \quad [086]$$

With the no speculative bubbles condition,

$$\lim_{t \rightarrow \infty} \left(\frac{1-\delta}{1+r} \right)^t Q_t = 0 \quad [087]$$

and after rearranging, we get

$$Q_t = \frac{1}{(1+r)} \sum_{s=1}^{\infty} \left(\frac{1-\delta}{1+r} \right)^{s-1} p_{t+s} \frac{\partial F}{\partial K_{t+s}} \quad [088]$$

Given [052], we can write equivalently

$$Q_t = \frac{1}{(1+r)} \sum_{s=1}^{\infty} \left(\frac{1-\delta}{1+r} \right)^{s-1} R_{t+s} \quad [089]$$

Condition [089] thus shows that at optimum, the marginal cost of capital must be equal to its marginal revenue, which is the discounted sum of future flows of income it will generate, that is, of values of marginal product R_{t+s} ¹⁰. These flows diminish in time as the capital depreciates; whence, attrition factor $(1-\delta)^{s-1}$.

1.2.5 Investment demand with stationary expectations

The equation for the marginal cost of new capital

$$Q_t = \frac{\partial}{\partial I_t} \left[q_t I_t \left(1 + \frac{\gamma}{2} I_t \right) \right] = q_t (1 + \gamma I_t) \quad [065]$$

is equivalent to

$$I_t = \frac{1}{\gamma} \left(\frac{Q_t}{q_t} - 1 \right) \quad [091]$$

That relation can be interpreted as an investment demand, but equation

¹⁰ Since adjustment costs are independent of the quantity of capital already in place, there are no avoided costs, contrary to what can be seen in equation [252] of Appendix A1.

$$Q_t = \frac{1}{(1+r)} \sum_{s=1}^{\infty} \left(\frac{1-\delta}{1+r} \right)^{s-1} R_{t+s} \quad [089]$$

shows that Q_t depends on the future values of R_{t+s} , so that its value is unknown in period t , unless assumptions are made concerning R_{t+s} . Suppose that \tilde{R}_{t+s} , the expected value at time t of R_{t+s} , is constant (stationary expectations) :

$$\tilde{R}_{t+s} = R_t, \forall s \geq 0 \quad [054]$$

Then

$$Q_t = \frac{1}{(1+r)} \sum_{s=1}^{\infty} \left(\frac{1-\delta}{1+r} \right)^{s-1} \tilde{R}_{t+s} \quad [092]$$

$$Q_t = \frac{1}{(1+r)} \sum_{s=1}^{\infty} \left(\frac{1-\delta}{1+r} \right)^{s-1} R_t \quad [093]$$

That equation can be rewritten using geometric series formula :

$$\sum_{s=0}^{\infty} \left(\frac{1-\delta}{1+r} \right)^s = \left(\frac{1}{1 - \frac{1-\delta}{1+r}} \right) \quad [094]$$

$$\sum_{s=0}^{\infty} \left(\frac{1-\delta}{1+r} \right)^s = \frac{1}{\left(\frac{r+\delta}{1+r} \right)} \quad [095]$$

$$\sum_{s=0}^{\infty} \left(\frac{1-\delta}{1+r} \right)^s = \frac{1+r}{r+\delta} \quad [096]$$

Substitute [096] and [054] in [089], and there results

$$Q_t = \frac{1}{(r+\delta)} R_t \quad [097]$$

$$R_t = (r+\delta) Q_t \quad [098]$$

Substituting [097] into [091], we find the investment demand equation with stationary expectations, and adjustment costs of the form [062] :

$$I_t = \frac{1}{\gamma} \left(\frac{R_t}{(r+\delta)q_t} - 1 \right) \quad [099]$$

the denominator of R_t in [099] is equal to

$$\bar{u}_t = (r + \delta) q_t \quad [056]$$

It is the user cost of capital with stationary expectations and no adjustment costs, while equilibrium condition [018] reduces to [058] :

$$R_t = (r + \delta) q_t = \bar{u}_t \quad [058]$$

With myopic expectations [054], and in the absence of adjustment costs, $\gamma = 0$ and $Q_t = q_t$, so that [097] reduces to [058].

1.2.6 Tobin's « q » again

In equation [099], the ratio $\frac{R_t}{(r + \delta)q_t}$ can be interpreted as an approximation of Tobin's « q ».

Indeed, with myopic expectations, the present value of the income¹¹ expected from one unit of capital is

$$\sum_{\theta=1}^{\infty} \frac{(1-\delta)^{\theta-1}}{(1-r)^{\theta}} R_t = \frac{R_t}{1-\delta} \sum_{\theta=1}^{\infty} \left(\frac{1-\delta}{1+r} \right)^{\theta} = \frac{R_t}{(r+\delta)}$$

which can be interpreted as the market value of a unit of capital. So $\frac{R_t}{(r+\delta)q_t}$ is the ratio of the

market value of a unit of capital to its replacement cost q_t , *ignoring adjustment costs*. Investment demand [099] is a function of the discrepancy between Tobin's « q » without adjustment costs, and the value 1. The greater parameter γ , the greater the adjustment cost, and the smaller the fraction of the discrepancy that will be eliminated by the optimal investment.

1.2.7 Reflexions on the stationary expectations hypothesis

Is the stationary expectations hypothesis

$$\tilde{R}_{t+s} = R_t, \forall s \geq 0 \quad [054]$$

¹¹ Note that *income* is a different concept from the cash flow in terms of which Tobin's « q » was discussed in section 1.2.3 above.

compatible with the model? Let us see what it implies. First note that equation [097] says that, with stationary expectations as described in [054], the marginal cost of new capital expected in t for period $t+s$ is also constant :

$$\tilde{Q}_{t+s} = \frac{1}{(r+\delta)} \tilde{R}_{t+s} = \frac{1}{(r+\delta)} R_t \quad [100]$$

And investment demand equation [099] also says that, if the expected value of the replacement cost of capital \tilde{q}_{t+s} is constant (which would be consistent with stationary expectations relative to price p_t), then the volume of investment expected in the future is also constant, equal to investment in the current period :

$$\tilde{I}_{t+s} = \frac{1}{\gamma} \left(\frac{1}{\tilde{q}_{t+s}} \frac{1}{(r+\delta)} \tilde{R}_{t+s} - 1 \right) = \frac{1}{\gamma} \left(\frac{1}{q_t} \frac{1}{(r+\delta)} R_t - 1 \right) = I_t \quad [101]$$

There is no need to question the compatibility of relations [100] and [101] : relation [100] is an implication of [097]; as for [101], it is a consequence of [099], which also follows from [097] and the definition of Q_t [065].

Let us now take a closer look at the hypothesis of stationary expectations itself

$$\tilde{R}_{t+s} = R_t, \forall s \geq 0 \quad [054]$$

If we assume that expectations are stationary for price p_t too, then it is necessary that it be possible for $\frac{\partial F}{\partial K_t}$ to be constant in order that it be possible for

$$R_t = p_t \frac{\partial F}{\partial K_t} \quad [052]$$

to be constant. But can marginal product $\frac{\partial F}{\partial K_t}$ be constant when net investment is not zero and the quantity of capital K_t varies in time? For the expected stock of capital in each future period is given by

$$\tilde{K}_{t+s} = (1-\delta) \tilde{K}_{t+s-1} + \tilde{I}_{t+s-1} \quad [102]$$

$$\tilde{K}_{t+s} = (1-\delta) \left[(1-\delta) \tilde{K}_{t+s-2} + \tilde{I}_{t+s-2} \right] + \tilde{I}_{t+s-1} \quad [103]$$

which, in view of [101], leads to

$$\tilde{K}_{t+s} = (1-\delta)^s K_t + I_t \left(\sum_{\theta=1}^s (1-\delta)^{\theta-1} \right) \quad [104]$$

by recurrence. It is clear that it is indeed possible to adjust the quantity of labor L_t in each future period in such a way as to maintain the marginal product of capital $\frac{\partial F}{\partial K_t}$ constant, with a stock of capital defined by [104]. But is it optimal?

If we assume constant returns to scale, the production function is first-degree homogenous, and verifies Euler's condition

$$F(K_t, L_t) = \frac{\partial F}{\partial K_t} K_t + \frac{\partial F}{\partial L_t} L_t \quad [040]$$

From [040], we get

$$\frac{\partial F}{\partial K_t} = \frac{F(K_t, L_t)}{K_t} - \frac{L_t}{K_t} \frac{\partial F}{\partial L_t} \quad [105]$$

$$\frac{\partial F}{\partial K_t} = F\left(1, \frac{L_t}{K_t}\right) - \frac{L_t}{K_t} \frac{\partial F}{\partial L_t} \quad [106]$$

Given first-order condition

$$p_t \frac{\partial F}{\partial L_t} = w_t \quad [016]$$

after substituting, we obtain

$$\frac{\partial F}{\partial K_t} = F\left(1, \frac{L_t}{K_t}\right) - \frac{L_t}{K_t} \frac{w_t}{p_t} \quad [107]$$

With constant returns to scale, and under the stationary expectations hypothesis for p_t , w_t and Q_t , the marginal product of capital $\frac{\partial F}{\partial K_t}$ is therefore constant if the optimal ratio $\frac{L_t}{K_t}$ is too. Is it?

Let's go back to the first-order optimality conditions

$$p_t \frac{\partial F}{\partial L_t} = w_t \quad [016]$$

$$p_t \frac{\partial F}{\partial K_t} = (1+r)Q_{t-1} - (1-\delta)Q_t \quad [066]$$

Under stationary expectations hypothesis [054], the second condition becomes

$$Q_t = \frac{1}{(r + \delta)} R_t \quad [097]$$

that is, given definition [052],

$$p_t \frac{\partial F}{\partial K_t} = (r + \delta) Q_t \quad [108]$$

In Euler's condition

$$F(K_t, L_t) = \frac{\partial F}{\partial K_t} K_t + \frac{\partial F}{\partial L_t} L_t \quad [040]$$

substitute from [016] and [108] :

$$F(K_t, L_t) = \frac{(r + \delta) Q_t}{p_t} K_t + \frac{w_t}{p_t} L_t \quad [109]$$

$$F\left(1, \frac{L_t}{K_t}\right) = \frac{(r + \delta) Q_t}{p_t} + \frac{w_t}{p_t} \frac{L_t}{K_t} \quad [110]$$

With constant returns to scale, and under the hypothesis of stationary expectations for p_t , w_t and Q_t , the ratio $\frac{L_t}{K_t}$ which is optimal in period t is also optimal for all future periods.

We may conclude that the hypothesis of stationary expectations [054] does not introduce any contradictions into the model.

2. Applied models of investment demand

In this section, we survey investment demand models that are, closely or distantly, related to the theoretical model reviewed in the preceding chapter.

2.1 BOURGUIGNON, BRANSON AND DE MELO, J. (1989)

Let us first examine the formulation of Bourguignon, Branson and de Melo (1989), also used by Decreux (2003).

Let us denote the user cost of capital with stationary expectations as

$$\bar{u}_t = (r + \delta) q_t \quad [056]$$

Rewrite [099]

$$I_t = \frac{1}{\gamma} \left(\frac{R_t}{\bar{u}_t} - 1 \right) \quad [111]$$

The latter equation is of the same form as the theoretical investment demand function of the representative firm in an industry, as written by Bourguignon *et al.* (1989, p. 23, equation 4.21 in their document) :

$$I_t = a \left\{ \frac{p^n MP_k U}{q(\delta + J^F)} - 1 \right\} = a \left\{ \frac{B}{C} - 1 \right\} \geq 0 \quad [112]$$

where

p^n is the price of value added

MP_k is the marginal product of capital

U is the industry rate of capacity utilization

q is the price of capital goods

δ is the rate of depreciation

J^F is the « opportunity cost of credit » (equation 4.22 in their document)

and where investment is constrained to be non-negative.

The B and the C in Bourguignon *et al.* can be compared to R_t and \bar{u}_t in [111] respectively.

Investment will be positive if the ratio $\frac{R_t}{\bar{u}_t}$ is greater than 1. It is profitable to invest when the

value of the marginal product of capital is greater than its user cost.

But Bourguignon *et al.* (1989) add : « However, with this specification, the model exhibits extreme fluctuations to changes in the relative profitability of investment caused by interest rate or expectation changes. For this reason, real investment is given by the quadratic expression » (p. 28)

$$\frac{I_t}{K_t} = q\gamma_1 \left[\left(\frac{B}{C} \right)^2 + \gamma_2 \left(\frac{B}{C} \right) \right] \quad [113]$$

where

- γ_1 and γ_2 « are suitably selected parameters so that in equilibrium when $\left(\frac{B}{C} \right) = 1$, investment will be at a level which will ensure a rate of growth of net capital stock equal to g .

The elasticity of investment with respect to a change in profitability, $\frac{\partial I}{\partial \left(\frac{B}{C}\right)}$, evaluated at

$\left(\frac{B}{C}\right) = 1$ is equal to a predetermined value ϵ » (p. 28; also see figure 3, p. 30 in Bourguignon et al.).

That formulation deviates from theoretical formulation [112] in two aspects : (1) it is an equation of the rate of accumulation $\frac{I_t}{K_t}$, rather than of investment as such, and (2) it is a quadratic form

of the ratio $\frac{R_t}{\bar{u}_t}$, rather than a linear function of the difference $\frac{R_t}{\bar{u}_t} - 1$.

The first difference is easily justified if one remembers that the theoretical model concerns an individual firm. Speaking of the « representative firm » in a dynamic general equilibrium model, it is reasonable to think that investments will grow with the number of firms. And if one admits that the number of firms increases in proportion to the stock of capital, it is appropriate to write the aggregate investment demand function as

$$\frac{I_t}{K_t} = \frac{1}{\gamma} \left(\frac{R_t}{\bar{u}_t} - 1 \right) \quad [114]$$

As a matter of fact, Nickell's microeconomic investment function

$$I_t = \frac{1}{\gamma} \left(\frac{R_t}{\bar{u}_t} - 1 \right) \quad [111]$$

cannot generate a regular path, that is, a path where prices are constant and where quantities increase at the exogenous rate of growth of labor supply (demographic growth rate) : if $\frac{R_t}{\bar{u}_t}$ is constant, so is investment, and the only regular path that may result is a stationary state *strictu sensu*, without growth (otherwise, with a constant volume of investment, the rate of growth of capital falls).

As for the second difference, it seems to be *ad hoc*, without clear theoretical foundations. For that reason, one would normally prefer the theoretically exact formulation. Whence the need to understand what the authors mean exactly by « the model exhibits extreme fluctuations to changes in the relative profitability of investment caused by interest rate or expectation

changes ». Experiments conducted with a small scale model led us to determine that [114] implies a degree of investment demand elasticity that is simply too high for the model to be stable (we shall return to this point in 4.3).

2.2 JUNG AND THORBECKE (2001)

The Jung and Thorbecke (2001, equation 41) equation for investment into industry i is

$$\frac{INV_{it}}{K_{it}} = A_i \left(\frac{KINC_{it}}{PK_{it} K_{it} r_t} \right)^{\beta_i} \quad [115]$$

where

INV_{it} is investment into industry i ;

K_{it} is industry i 's capital stock;

$KINC_{it}$ is capital income;

PK_{it} is the price of the investment good in industry i ;

r_t is the interest rate, which plays the part of the discount rate;

A_i and β_i are parameters.

The value of elasticity β_i is set to 1¹².

It is obvious that the product $PK_{it} K_{it}$ is the replacement cost of capital. On the other hand, the

ratio $\frac{KINC_{it}}{r_t}$ is the present value, with discount rate r_t , of a perpetual flow of income of $KINC_{it}$

per period, beginning in period $t+1$: it can be taken as an approximation of the stock market

value of industry i . It follows that the ratio $\left(\frac{KINC_{it}}{PK_{it} K_{it} r_t} \right)$ can be interpreted as an approximation

of Tobin's q .

Formally, let us compare the ratio $\left(\frac{KINC_{it}}{PK_{it} K_{it} r_t} \right)$ with Tobin's q in the basic discrete-time model

developed in 1.1.3 :

$$\frac{\sum_{s=1}^{\infty} \frac{1}{(1+r)^s} [p_{t+s} F(K_{t+s}, L_{t+s}) - w_{t+s} L_{t+s} - q_{t+s} I_{t+s}]}{q_t K_{t+1}} = 1 \quad [043]$$

¹² Private communication of Nabil Annabi with Hong Sang Jung.

If, in [043], one assumes stationary expectations and no future investments, the numerator becomes

$$\sum_{s=1}^{\infty} \frac{1}{(1+r)^s} [p_t F(K_t, L_t) - w_t L_t] = \frac{1}{r} [p_t F(K_t, L_t) - w_t L_t] \quad [116]$$

where $[p_t F(K_t, L_t) - w_t L_t]$ is capital income in the current period, that is, equivalent in our notation to Jung and Thorbecke's $KINC_{it}$ (2001, equation 22 in their paper). Thus, the numerator of the right-hand-side of [115] is not quite the same as the numerator of Tobin's q ratio. Ignoring the term $q_{t+s} I_{t+s}$ amounts more or less to ignoring depreciation, since there is not even replacement investment. That is the reason for speaking of an approximation.

So in Jung and Thorbecke's (2001) model, the rate of investment $\frac{INV_{it}}{K_{it}}$ is a constant-elasticity function of Tobin's q , but of a truncated version of the latter, because without investment, even for replacement, no account is taken of depreciation. Practically speaking, given the functional form, replacement investment is implicit in the constant A_i . For, when the ratio $\left(\frac{KINC_{it}}{PK_{it} K_{it} r_t} \right)$ is

equal to 1, equation [115] becomes

$$\frac{INV_{it}}{K_{it}} = A_i \quad [117]$$

The constant A_i is therefore the equilibrium level of investment, that is, the one that prevails when Tobin's (truncated) q is equal to 1.

It is easy, however, to reformulate Jung and Thorbecke's model to take account of depreciation.

It is sufficient to replace the ratio $\frac{KINC_{it}}{r_t}$ by $\frac{KINC_{it}}{r_t + \delta}$. Indeed, the latter is equal to

$$\sum_{s=1}^{\infty} \frac{(1-\delta)^{s-1}}{(1+r)^s} KINC_{it}$$

where the flow of future income declines in step with the depreciation of capital. We then obtain

$$\frac{INV_{it}}{K_{it}} = A_i \left(\frac{KINC_{it}}{PK_{it} K_{it} (r_t + \delta)} \right)^{\beta_i} \quad [118]$$

2.3 AGÉNOR (2003)

That author presents a real version of the World Bank's IMMPA (« Integrated Macroeconomic Model for Poverty Analysis »), used for analyzing the effects of economic policies on poverty.

Agénor (2003) formulates the following equation :

$$\frac{I}{K} = z \left[\frac{\text{profit}}{PK \left(i^* + \delta - \frac{\Delta PK}{PK} \right)} \right]^\sigma \quad [119]$$

where

I is the volume of investment;

K is the stock of capital;

PK is the price of the investment good;

i^* is the world interest rate;

δ is the rate of depreciation of capital;

z is a scale parameter;

σ is the elasticity of investment.

That model is similar to Jung and Thorbecke's (2001), with the exception that Agénor takes into account depreciation and inflation.

2.4 FARGEIX AND SADOULET (1994)

Fargeix and Sadoulet's (1994) equation for investment into industry i is :

$$\frac{I_{it}}{K_{it}} = B_i \left(\frac{KINC_{it} (1 + \pi_t)}{PK_{it} K_{it} (1 + rd_t)} \right)^{\varepsilon_i} \quad [120]$$

where

I_{it} is investment into industry i ;

K_{it} is the stock of capital of industry i ;

B_i is a scale parameter;

$KINC_{it}$ is capital income;

PK_{it} is the price of the investment good in industry i ;

π_t is the period t rate of inflation;

rd_t is the period t discount rate.

That equation resembles Jung and Thorbecke's (2001), except that the ratio $\frac{1+\pi_t}{1+rd_t}$ is substituted for $\frac{1}{r_t}$. The ratio $\frac{KINC_{it}(1+\pi_t)}{1+rd_t}$ is the present value at time t of an income of $KINC_{it}(1+\pi_t)$ received in period $t+1$, discounted at rate rd_t . The ratio $\left(\frac{KINC_{it}(1+\pi_t)}{PK_{it}K_{it}(1+rd_t)}\right)$ is therefore related to Tobin's q . Related, but not identical to it : the numerator of Tobin's q would rather be

$$\sum_{s=1}^{\infty} \left(\frac{1+\pi_t}{1+rd_t}\right)^{t+s} KINC_t = \frac{1+\pi_t}{rd_t - \pi_t} KINC_t \quad [121]$$

providing $rd_t > \pi_t$. The ratio $\left(\frac{KINC_{it}(1+\pi_t)}{PK_{it}K_{it}(1+rd_t)}\right)$ could be called a truncated q , or a single-future-period q .

Fargeix and Sadoulet's (1994) formula is thus similar in practice to Jung and Thorbecke's, but its relationship to Tobin's theoretical q concept is less rigorous.

2.5 COLLANGE (1993)

Collange's (1993) investment function is

$$\frac{I_j}{K_j} = B_j \left(\frac{RK_j}{PK_j K_j}\right)^{\sigma_1} \left(\frac{J_e}{1+PINFL}\right)^{\sigma_2} \left(\frac{Autofin}{Autofin0}\right)^{\sigma_3} \quad [122]$$

where

B_j is a scale parameter;

RK_j is capital income;

PK_j is the replacement cost of capital;

K_j is the stock of capital;

J_e is the cost of borrowing;

$PINFL$ is the inflation rate;

$Autofin$ is businesses' self-financing capability;

$Autofin0$ is businesses' base-year self-financing capability;

$\sigma_1 > 0$, $\sigma_2 < 0$ and $\sigma_3 > 0$ are investment elasticities.

That specification too is close to that of Jung and Thorbecke (2001). If $\sigma_2 = -\sigma_1$, the ratio $\left(\frac{J_e}{1+PINFL}\right)$ plays the same role in Collange's (1993) model as discount rate r_t in Jung and Thorbecke's (2001).

But, in the author's own words, it is an private investment demand function determined « in an *ad hoc* way » (Collange, 1993, p. 24), which, moreover, seeks to take into account the importance of self-financing in the Ivory Coast context.

2.6 INVESTMENT DEMAND IN CGE MODELS

The investment demand models presented here distribute investment among industries, given the amount of savings. Investment-savings equilibrium is guaranteed by the interest rate, which plays the part of a discount rate and enters the determination of the user cost of capital : since the rate of interest is the same for all industries, it follows that the rate of return for all capital owners is uniform across industries. The balancing of the sum of investment demands and available income is realized, either by adjusting that interest rate (endogenous interest rate), either by adjusting the current account balance (endogenous Rest-of-the-World savings), depending on model closure.

It should be noted that, in the first case, the endogenous interest rate may have no other role than to « ration » available savings. It does not necessarily fully play the part of a price, since the remuneration of newly invested capital does not take place in the current period, and that, when it does take place, in subsequent periods, it is determined independently of the current period discount rate. Nonetheless, we shall see that that interest rate can create a link between the issue of investment by destination, and the issues of savings and debt (second and third part).

3. Non-demand models of the distribution of investment by destination

Investment demand models are not the only possible approach to determining investment by destination. There exist a number of models that determine industry shares in the investment total. Some of them are purely *ad hoc*. Others are more or less closely related to a theory of investment supply. The first section of the present chapter sketches a model of investment supply. The following sections present the different models found in the literature.

3.1 A SKETCH OF AN INVESTMENT SUPPLY MODEL

3.1.1 A simplified model of the investor capitalist

The capitalist's goal is to maximize his/her net worth. A unit (volume) of investment is presumed to earn an amount of rs_i per period, forever (stationary expectations). With a discount rate of TIN , the present value of the income flow generated by a quantity IND_i of capital is equal to

$$\sum_{t=0}^{\infty} \left(\frac{rs_i IND_i}{(1+TIN)^t} \right) = \frac{rs_i IND_i}{TIN} \quad [123]$$

where depreciation is ignored to simplify.

Acquisition of that quantity of capital costs $PK_i IND_i$. Thus, the investor-capitalist's problem is to distribute his/her investment budget between possibilities i in such a way as to maximize the present value of his/her net worth :

$$MAX \sum_i \left(\frac{rs_i}{TIN} - PK_i \right) IND_i \quad [124]$$

$$\text{subject to } \sum_i PK_i IND_i \leq IT \quad [125]$$

where

rs_i is the rental rate of capital i ;

TIN is the interest rate (which acts as discount rate);

PK_i is the price (replacement cost) of capital i ;

IND_i is the quantity of investment into industry i (increase in i 's capital);

IT is the investment budget.

Form the Lagrangian

$$\Lambda = \sum_i \left(\frac{rs_i}{TIN} - PK_i \right) IND_i + \lambda \left(IT - \sum_i PK_i IND_i \right) \quad [126]$$

And the Kuhn and Tucker¹³ conditions are

$$\frac{\partial \Lambda}{\partial IND_i} = \left(\frac{rs_i}{TIN} - PK_i \right) - \lambda PK_i \leq 0 \quad [127]$$

¹³ They are to be preferred to first-order conditions here, because, in general, classical optimum first-order conditions cannot be simultaneously satisfied, as we shall see in a moment.

$$IND_i \geq 0 \text{ (non-negativity constraint)} \quad [128]$$

$$IND_i \left[\left(\frac{rs_i}{TIN} - PK_i \right) - \lambda PK_i \right] = 0 \text{ (orthogonality constraint)} \quad [129]$$

$$\left(IT - \sum_i PK_i IND_i \right) \geq 0 \quad [130]$$

$$\lambda \geq 0 \text{ (non-negativity constraint)} \quad [131]$$

$$\lambda \left(IT - \sum_i PK_i IND_i \right) = 0 \text{ (orthogonality constraint)} \quad [132]$$

Condition [127] can be written

$$\frac{rs_i}{TIN} - (1 + \lambda) PK_i \leq 0 \quad [133]$$

$$\frac{rs_i}{TIN} \leq (1 + \lambda) PK_i \quad [134]$$

$$\left(\frac{rs_i / TIN}{PK_i} \right) \leq (1 + \lambda) \quad [135]$$

That condition may be verified with strict equality for all i only if all the ratios $\frac{rs_i}{PK_i}$ are equal.

Otherwise, orthogonality constraint [129] implies that $IND_i = 0$ for all i such that

$$\left(\frac{rs_i / TIN}{PK_i} \right) < (1 + \lambda) \text{ (strict inequality)} \quad [136]$$

That means that investment is null in any industry i where the ratio $\frac{rs_i}{PK_i}$ is less than the highest

observed value in the set of all industries. It follows that, if all the ratios $\frac{rs_i}{PK_i}$ are different, the

investor-capitalist will put all of his/her savings IT in a single possibility.

It should be noted that the left-hand-side member of [135] is analog to Tobin's q . Indeed, the latter is defined as

$$\frac{\text{Stock market valuation of the firm}}{\text{Replacement cost of its capital}}$$

Now, the stock market valuation of the firm, in Tobin's investment theory, is the result of investors' expectations relative to the present value of its profits. The equivalent in our context is $\left(\frac{rs_i}{TIN}\right)$, the present value of the income generated by one unit of capital. And, naturally, PK_i is the replacement cost of a unit of capital.

3.1.2 Random utility and multinomial logit

Here we present a random utility model which makes it possible to generalize the simplified model developed in the preceding subsection.

Denote

$$v_i = \left(\frac{rs_i}{TIN} - PK_i\right) \quad [137]$$

That v_i is the coefficient of IND_i in objective function [124].

If one supposes that all investors foresee that the current value of v_i will persist in the future (stationary expectations), we have seen that all investments will be allocated to the industry i with the highest v_i .

How can investment behavior be pictured as rational without all investment being concentrated in a single industry? The random utility model offers a possibility. According to that discrete choice model, each individual rationally chooses that possibility which yields the greatest utility. But the utility of a given possibility for a given individual is not deterministic.

In the present context, it is reasonable to believe that investors are not perfectly unanimous in their expectations. To represent such a dispersion of expectations, let's write the utility of investment i for investor n as

$$U_{in} = \beta_i v_i + \varepsilon_{in} \quad [138]$$

where

$$v_i = \left(\frac{rs_i}{TIN} - PK_i\right) \text{ is the net present value of an investment of one unit of capital into industry } i$$

under the assumption of stationary expectations;

rs_i is the rental rate of capital in industry i ;

β_i is the parameter describing the sensitivity of investors to the v_i 's;

$\beta_i v_j$ is therefore the systematic part of utility; parameter β_i is necessary to define the relative weight of systematic utility relative to the random term;
 ε_{in} is a random term.

Given that individual expectations depend on many of factors, several of which are unobservable, it is suitable to represent these variations by means of a random term.

In the random utility model, therefore, since subject n rationally chooses the possibility which yields the greatest utility to himself/herself, the probability that subject n choose option i is equal to the probability that option i have a greater utility for him/her than any other option available.

Ultimately, the specific form of the model obviously depends on which hypotheses are made relative to the probability distribution of the random terms ε_{in} . Domencich and McFadden (1975, chap. 4) explore several possibilities. The one that leads to the multinomial logit model is that the random terms follow a Gumbel distribution :

$$\text{Prob}[E \leq \varepsilon] = F(\varepsilon) = \exp[-e^{-\mu(\varepsilon - \eta)}] \quad [139]$$

where $\mu > 0$ is a scale parameter, and η , a position parameter.

So we make the following assumptions :

- the random terms are independent;
- they are identically distributed;
- their distributions are Gumbel distributions :

$$E_{in} \sim G(\eta, \mu), \text{ for any combination } i, n.$$

Those hypotheses lead to the standard multinomial logit form

$$\text{Pr}_n(i) = \frac{\exp(\beta_i v_i)}{\sum_j \exp(\beta_j v_j)} \quad [140]$$

With a large number of investors n , all identical save for the value of the random terms, the probability $\text{Pr}_n(i)$ yields the distribution of investments among industries.

To our knowledge, the multinomial logit model has not been applied to the distribution of investment among industries in a CGE model¹⁴. Yet, among the models surveyed below, some are definitely akin to the multinomial logit model.

¹⁴ According to Thissen (1999), however, Easterly (1990) uses a multinomial logit model for the portfolio allocation of households' savings.

3.2 BEGHIN, DESSUS, ROLAND AND MENSBRUGGHE (1996)

Beghin, Dessus, Roland and Mensbrugghe (1996), and Mensbrugghe (2003), have developed two versions of their model, one static, and the other recursive dynamic.

3.2.1 Static single vintage framework

That version of the model is static, and not recursive dynamic. We nonetheless report on it, because the way in which the distribution of capital among industries could equally apply to the distribution of investments in each period in a recursive dynamic model. Note that a similar specification can be used to represent capital mobility in a static model (Decaluwé *et al.*, 2005).

In that version of the model, capital mobility between industries is represented by a CET function :

$$K^S = \left[\sum_i \left(\gamma_i^k \right)^{\omega^K} (KS_i)^{\frac{1+\omega^K}{\omega^K}} \right]^{\frac{\omega^K}{1+\omega^K}} \quad [141]$$

where

K^S is the total supply of capital;

KS_i is capital supply to industry i ;

ω^K is the elasticity of transformation of capital

γ_i^k is a share parameter.

Holders of capital maximize their total income

$$\sum_i R_i KS_i, \text{ subject to the CET transformation function}$$

where

R_i is the rental rate of capital in industry i ¹⁵.

The supply of capital to industry i is then

$$KS_i^S = \gamma_i^k \left(\frac{R_i}{TR} \right)^{\omega^K} K^S \text{ si } 0 \leq \omega^K < \infty \quad [142]$$

¹⁵ Note the distinction between « rental rate » and « rate of return ». The latter refers to the ratio of capital income to the amount invested, while the former is the amount of income paid to the owner per unit of capital used.

where

TR is the aggregate rental rate of capital.

If elasticity is infinite, supply is infinitely elastic to aggregate rental rate TR :

$$R_i = TR \text{ if } \omega^K = \infty \quad [143]$$

The dual to the transformation function is the rental rate aggregation function :

$$TR = \left[\sum_i \gamma_i^k (R_i)^{1+\omega^K} \right]^{\frac{1}{1+\omega^K}} \text{ si } 0 \leq \omega^K < \infty \quad [144]$$

If the transformation elasticity is infinite, equation [144] is replaced by the constraint

$$\sum_i K_i^d = K^s \text{ if } \omega^K = \infty \quad [145]$$

3.2.2 Multiple vintage framework

In the multiple vintage framework of Beghin, Dessus, Roland and Mensbrugghe (1996), and of Mensbrugghe (2003), there is no proper *investment* demand. There is only demand and supply of *capital*, with perfect mobility for new capital, and partial mobility for old capital. And the aggregate supply of capital is independent of capital income. For old capital is inherited from the preceding period; as for new capital, it is simply the ratio of the preceding period's savings to the aggregate price of investment in the same period, while foreign savings are exogenous, and household savings are a constant fraction of supernumerary income.

We refer here to the simplified version of the dynamic model presented by Mensbrugghe (2003)¹⁶. In each period, the aggregate supply of capital consists of capital inherited from the preceding period, adjusted for depreciation, and new capital. The latter is the preceding period's investment. But the new capital has not been distributed among industries in the preceding period; that distribution occurs in the *current* period : new capital created in the preceding period is perfectly mobile among industries in the current period. There are distinct demands for old and new capital, in each industry ; the demand for capital is derived from the production functions. The rental rate of capital is endogenous, the same for all industries, *except* for those experiencing negative growth, that is, those whose demand does not exceed installed capacity

¹⁶ In Mensbrugghe's (2003) simplified version, there is no government, and no business savings. But even in the more elaborate version of Beghin *et al.* (1996), the aggregate supply of capital remains independent of the rental rate of capital.

inherited from the preceding period. Industries experiencing negative growth release capital; capital retained in a shrinking industry is a constant elasticity function of the ratio

$$\frac{R_{i,t}^{old} / R_{i,t}^{new}}{R_{i,t-1}^{old} / R_{i,t-1}^{new}} \quad [146]$$

where $R_{i,t}^{old}$ is the rental rate of old capital at time t in industry i , and $R_{i,t}^{new}$ is the rental rate of new capital at time t in industry i .

Old capital released by receding industries is melted into new capital, from which it becomes indistinguishable. So there is a form of partial mobility of old capital.

It can be seen that, as we had stated, there is no *investment* demand in that model. There is only demand and supply of *capital*, with perfect mobility for new capital, and partial mobility for old capital. The reason the model is devoid of investment demand is the absence of an installation lag. Of course, one could imagine a model with both an installation lag for new capital and a multiple vintage technology. But inspiration for the specification of investment demand would have to be found elsewhere than in Beghin *et al.*

3.3 THE *MIRAGE* MODEL OF BCHIR, DECREUX, GUÉRIN AND JEAN (2002)

The investment distribution function in the *MIRAGE* model of Bchir, Decreux, Guérin and Jean (2002) belongs to the family of gravity models, widely used in regional science and quantitative geography. So we shall present the gravity model before examining the *MIRAGE* model itself.

3.3.1 The gravity model

3.3.1.1 Model statement

Newton's Law of gravity is

$$F = \frac{Gm_1m_2}{d^2} \quad [147]$$

where

F is the attraction force of gravity;

m_1 and m_2 are the respective masses of two bodies;

d is the distance between them;

G is the gravitational constant¹⁷.

By analogy with the mechanics of physics, the following model has been proposed for origin-destination flows¹⁸

$$N_{od} = \frac{G_{od} O_o D_d}{f(d_{od})} \quad [148]$$

where

N_{od} is the flow from origin o to destination d ;

O_o is supply at origin o ;

D_d is demand in destination d ;

d_{od} is the distance between origin o and destination d ;

$f(d_{od})$ is the friction of distance;

G_{od} is a calibration constant.

That model is formally different from Newton's Law in that constant G_{od} is specific to every origin-destination pair. That is imposed by equilibrium constraints

$$\sum_o N_{od} = D_d \sum_o \frac{G_{od} O_o}{f(d_{od})} = D_d \text{ for any destination } d \quad [149]$$

$$\sum_d N_{od} = O_o \sum_d \frac{G_{od} D_d}{f(d_{od})} = O_o \text{ for any origin } o \quad [150]$$

3.3.1.2 Application to investment by destination

We shall examine that formulation in the context of a model with a single origin. Concretely, that could correspond to a model of a single country or region, or a model with several regions in which savings are consolidated, as savings in Quebec and in the Rest-of-Canada are in the Ministère des Finances du Québec CGE model (MÉGFQ) (Decaluwé et al., 2005). In the *MIRAGE* model however, transborder investment flows, with multiple origins and destinations, are explicitly represented.

¹⁷ The gravitational constant is equal to $6,67259 \times 10^{-11} \text{ Nm}^2 \text{ kg}^{-2}$.

¹⁸ See, among others, the presentation by Wilson (1970b, p. 15 and following).

Let us nonetheless concentrate for the moment on the single origin case. Transposing the gravity model, investment expenditures by destination are represented as

$$PK_{k,i,rg,t} IND_{k,i,rg,t} = \frac{G_{k,i,rg,t} D_{k,i,rg,t}}{f(d_{k,i,rg,t})} \quad [151]$$

where

$PK_{k,i,rg,t}$ is the replacement cost of type k capital in industry i of region rg at time t ,

$IND_{k,i,rg,t}$ is type k investment into industry i of region rg at time t ,

$G_{k,i,rg,t}$ is a balancing variable.

Balancing variable $G_{k,i,rg,t}$ must verify the equilibrium constraint

$$\sum_k \sum_i \sum_{rg} PK_{k,i,rg,t} IND_{k,i,rg,t} = \sum_k \sum_i \sum_{rg} \frac{G_{k,i,rg,t} D_{k,i,rg,t}}{f(d_{k,i,rg,t})} = IT_t \quad [152]$$

where IT_t is total investment in period t .

We shall return later on to balancing variable $G_{k,i,rg,t}$. But, in a CGE model, what would correspond to $D_{k,i,rg,t}$ and $f(d_{k,i,rg,t})$?

Remember that, in the gravity model, so-called « demand » D_d does not so much represent demand to be satisfied as the « force of attraction » of a destination, its potential market. By analogy, that could correspond, in a CGE model, to installed capacity, that is, to the stock of capital, evaluated at replacement cost $PK_{k,i,rg,t} KS_{k,i,rg,t}$. As for the « friction of distance, it would be logical that it be an inverse function of the rental rate of capital $rs_{k,i,rg,t}$; for example

$$f(d_{k,i,rg,t}) = e^{-\alpha rs_{k,i,rg,t}}, \text{ where } \alpha \text{ is a free parameter.} \quad [153]$$

The model thus becomes

$$PK_{k,i,rg,t} IND_{k,i,rg,t} = G_{k,i,rg,t} e^{\alpha rs_{k,i,rg,t}} PK_{k,i,rg,t} KS_{k,i,rg,t} \quad [154]$$

with the constraint

$$\sum_k \sum_i \sum_{rg} PK_{k,i,rg,t} IND_{k,i,rg,t} = \sum_k \sum_i \sum_{rg} \left(G_{k,i,rg,t} e^{\alpha rs_{k,i,rg,t}} PK_{k,i,rg,t} KS_{k,i,rg,t} \right) = IT_t \quad [155]$$

where

$KS_{k,i,rg,t}$ is the stock of type k capital in industry i of region rg at time t ,

$rs_{k,i,rg,t}$ is the rental rate received by type k capital owners in industry i of region rg at time t ,

The balancing constraint will be verified if

$$G_{k,i,rg,t} = \frac{A_{k,i,rg} IT_t}{\sum_{kj} \sum_j \sum_{rgj} \left(e^{\alpha rs_{kj,j,rgj,t}} A_{kj,j,rgj} PK_{kj,j,rgj,t} KS_{kj,j,rgj,t} \right)} \quad [156]$$

where the $A_{k,i,rg}$ are calibrated constants (they are constant over time, contrary to the $G_{k,i,rg,t}$).

For the model to reproduce base year observations, it is necessary, at $t = 0$, that balancing variable $G_{k,i,rg,t}$ respect

$$PK_{k,i,rg,0} IND_{k,i,rg,0} = G_{k,i,rg,0} e^{\alpha rs_{k,i,rg,0}} PK_{k,i,rg,0} KS_{k,i,rg,0} \quad [157]$$

that is

$$PK_{k,i,rg,0} IND_{k,i,rg,0} = \frac{A_{k,i,rg} IT_0 e^{\alpha rs_{k,i,rg,0}} PK_{k,i,rg,0} KS_{k,i,rg,0}}{\sum_{kj} \sum_j \sum_{rgj} \left(e^{\alpha rs_{kj,j,rgj,0}} A_{kj,j,rgj,0} PK_{kj,j,rgj,0} PK_{kj,0} KS_{kj,j,rgj,0} \right)} \quad [158]$$

where the $A_{k,i,rg}$ are defined to a factor of proportionality. Indeed,

$$\begin{aligned} & \frac{A_{k,i,rg} IT_0 e^{\alpha rs_{k,i,rg,0}} PK_{k,i,rg,0} KS_{k,i,rg,0}}{\sum_{kj} \sum_j \sum_{rgj} \left(e^{\alpha rs_{kj,j,rgj,0}} A_{kj,j,rgj,0} PK_{kj,j,rgj,0} KS_{kj,j,rgj,0} \right)} \\ &= \frac{\lambda A_{k,i,rg} IT_0 e^{\alpha rs_{k,i,rg,0}} PK_{k,i,rg,0} KS_{k,i,rg,0}}{\lambda \sum_{kj} \sum_j \sum_{rgj} \left(e^{\alpha rs_{kj,j,rgj,0}} A_{kj,j,rgj,0} PK_{kj,j,rgj,0} KS_{kj,j,rgj,0} \right)} \\ &= \frac{\lambda A_{k,i,rg} IT_0 e^{\alpha rs_{k,i,rg,0}} PK_{k,i,rg,0} KS_{k,i,rg,0}}{\sum_{kj} \sum_j \sum_{rgj} \left(e^{\alpha rs_{kj,j,rgj,0}} \lambda A_{kj,j,rgj,0} PK_{kj,j,rgj,0} KS_{kj,j,rgj,0} \right)} \end{aligned} \quad [159]$$

So λ can be set so that

$$\sum_{kj} \sum_j \sum_{rgj} \left(e^{\alpha rs_{kj,j,rgj,0}} \lambda A_{kj,j,rgj,0} PK_{kj,j,rgj,0} KS_{kj,j,rgj,0} \right) = 1 \quad [160]$$

Equation [158] then amounts to

$$PK_{k,i,rg,0} IND_{k,i,rg,0} = A_{k,i,rg} IT_0 e^{\alpha rs_{k,i,rg,0}} PK_{k,i,rg,0} KS_{k,i,rg,0} \quad [161]$$

It follows that

$$A_{k,i,rg} = \frac{PK_{k,i,rg,0} IND_{k,i,rg,0}}{IT_0 e^{\alpha rs_{k,i,rg,0}} PK_{k,i,rg,0} KS_{k,i,rg,0}} \quad [162]$$

$$A_{k,i,rg} = \frac{IND_{k,i,rg,0}}{IT_0 e^{\alpha rs_{k,i,rg,0}} KS_{k,i,rg,0}} \quad [163]$$

So the $A_{k,i,rg}$ are calibrated. Next, substitute [156] into [154] and there results

$$PK_{k,i,rg,t} IND_{k,i,rg,t} = \frac{A_{k,i,rg} IT_t e^{\alpha rs_{k,i,rg,t}} PK_{k,i,rg,t} KS_{k,i,rg,t}}{\sum_{kj} \sum_j \sum_{rgj} \left(e^{\alpha rs_{kj,j,rgj,t}} A_{kj,j,rgj,t} PK_{kj,j,rgj,t} KS_{kj,j,rgj,t} \right)} \quad [164]$$

$$IND_{k,i,rg,t} = \frac{A_{k,i,rg} IT_t e^{\alpha rs_{k,i,rg,t}} KS_{k,i,rg,t}}{\sum_{kj} \sum_j \sum_{rgj} \left(e^{\alpha rs_{kj,j,rgj,t}} A_{kj,j,rgj,t} PK_{kj,j,rgj,t} KS_{kj,j,rgj,t} \right)} \quad [165]$$

$$\frac{IND_{k,i,rg,t}}{IT_t} = \frac{A_{k,i,rg} e^{\alpha rs_{k,i,rg,t}} KS_{k,i,rg,t}}{\sum_{kj} \sum_j \sum_{rgj} \left(e^{\alpha rs_{kj,j,rgj,t}} A_{kj,j,rgj,t} PK_{kj,j,rgj,t} KS_{kj,j,rgj,t} \right)} \quad [166]$$

We shall see that that form of the gravity model is the same as the investment distribution function in the *MIRAGE* model of Bchir *et al.* (2002). But one must admin that we have yet to give that specification a solid theoretical foundation : instead, it is derived from an intuitive analogy. We shall conclude however, at the end of 3.3.2, that that gravity model can be related to the multinomial logit model developed in 3.1.2.

3.3.2 The *MIRAGE* model of Bchir, Decreux, Guérin and Jean (2002)

The investment distribution function of Bchir, Decreux, Guérin and Jean (2002, p.119) is

$$\frac{PK_{s,irs}}{S_r} = \frac{A_{irs} PK_s K_{irs} e^{\alpha wk_{is}}}{\sum_{j,z} A_{jrz} PK_z K_{jrz} e^{\alpha wk_{jz}}} \quad [167]$$

where

PK_s is the price of the investment good in destination country s ¹⁹;

I_{irs} is the investment flow from country r to industry i in country s ;

S_r is total investment expenditures by country r ;

A_{irs} is a calibrated parameter;

K_{irs} is the stock of capital of industry i in country s owned by country r ;

wk_{is} is the return rate of capital of industry i in country s .

In the GAMS code of the *MIRAGE* model, the investment function is written as

$$I_{irs} = B_r A_{irs} K_{irs} e^{\alpha wk_{is}} \quad [167.1]$$

with variable B_r being determined by investment budget constraint

$$\sum_{i,s} PK_s I_{irs} = S_r \quad [167.2]$$

The investment budget constraint implies

$$\sum_{i,s} PK_s I_{irs} = \sum_{i,s} PK_s B_r A_{irs} K_{irs} e^{\alpha wk_{is}} = S_r \quad [167.3]$$

$$B_r = \frac{S_r}{\sum_{i,s} PK_s A_{irs} K_{irs} e^{\alpha wk_{is}}}, \text{ or, equivalently, } B_r = \frac{S_r}{\sum_{j,z} A_{jrz} PK_z K_{jrz} e^{\alpha wk_{jz}}} \quad [167.4]$$

Therefore, from [167.1],

$$I_{irs} = S_r \frac{A_{irs} K_{irs} e^{\alpha wk_{is}}}{\sum_{j,z} A_{jrz} PK_z K_{jrz} e^{\alpha wk_{jz}}} \quad [167.5]$$

$$I_{irs} = \frac{S_r}{PK_s} \frac{A_{irs} PK_s K_{irs} e^{\alpha wk_{is}}}{\sum_{j,z} A_{jrz} PK_z K_{jrz} e^{\alpha wk_{jz}}} \quad [167.6]$$

which is equivalent to [167].

After transposing [167] to the case of a single origin, and following the notation in [155], we have

¹⁹ In *MIRAGE*, the price of the investment good varies only with respect to destination country, but is constant across industries.

$$\frac{PK_{k,i,rg,t} IND_{k,i,rg,t}}{IT_t} = \frac{A_{k,i,rg} PK_{k,i,rg,t} KS_{k,i,rg,t} e^{\alpha rs_{k,i,rg,t}}}{\sum_{kj} \sum_j \sum_{rgj} A_{kj,j,rgj} PK_{kj,j,rgj,t} KS_{kj,j,rgj,t} e^{\alpha rs_{kj,j,rgj,t}}} \quad [168]$$

or, equivalently,

$$\frac{IND_{k,i,rg,t}}{IT_t} = \frac{A_{k,i,rg} KS_{k,i,rg,t} e^{\alpha rs_{k,i,rg,t}}}{\sum_{kj} \sum_j \sum_{rgj} A_{kj,j,rgj} PK_{kj,j,rgj,t} KS_{kj,j,rgj,t} e^{\alpha rs_{kj,j,rgj,t}}} \quad [169]$$

where $A_{k,i,rg}$ is a calibrated parameter, and α a free one. Indeed, it is easy to recognize the gravity model of equation [166].

Note that, in the absence of variation between rental rates (or, equivalently, if $\alpha = 0$), we have

$$\frac{PK_{k,t} IND_{k,i,rg,t}}{IT_t} = \frac{A_{k,i,rg} PK_{k,t} KS_{k,i,rg,t}}{\sum_{kj} \sum_j \sum_{rgj} A_{kj,j,rgj} PK_{kj,j,rgj,t} KS_{kj,j,rgj,t}} \quad [170]$$

The authors write their model (equation 3 in their paper) as :

$$IND_{k,i,rg,t} = B_t A_{k,i,rg} KS_{k,i,rg,t} e^{\alpha rs_{k,i,rg,t}} \quad [171]$$

$$\sum_k \sum_i \sum_{rg} PK_{k,t} IND_{k,i,rg,t} = IT_t \quad [172]$$

They define

$$R_t = \frac{1}{\alpha} \ln \left[\sum_k \sum_i \sum_{rg} \left(\frac{A_{k,i,rg} PK_{k,i,rg,t} KS_{k,i,rg,t}}{\sum_{kj} \sum_j \sum_{rgj} A_{kj,j,rgj} PK_{kj,j,rgj,t} KS_{kj,j,rgj,t}} e^{\alpha rs_{k,i,rg,t}} \right) \right] \quad [173]$$

or rather

$$e^{\alpha R_t} = \sum_k \sum_i \sum_{rg} \left(\frac{A_{k,i,rg} PK_{k,i,rg,t} KS_{k,i,rg,t}}{\sum_{kj} \sum_j \sum_{rgj} A_{kj,j,rgj} PK_{kj,j,rgj,t} KS_{kj,j,rgj,t}} e^{\alpha rs_{k,i,rg,t}} \right) \quad [174]$$

where the right-hand side is a weighed sum of the $e^{\alpha rs_{k,i,rg,t}}$, the weights being equal to the shares, as per [170], that would prevail in the absence of variation in the rental rates (or, equivalently, if $\alpha = 0$). Therefore

$$e^{\alpha R_t} \sum_{kj} \sum_j \sum_{rgj} A_{kj,j,rgj} PK_{kj,j,rgj,t} KS_{kj,j,rgj,t} = \sum_k \sum_i \sum_{rg} A_{k,i,rg} PK_{k,i,rg,t} KS_{k,i,rg,t} e^{\alpha rs_{k,i,rg,t}} \quad [175]$$

and the investment distribution equation can be written

$$\frac{PK_{k,i,rg,t} IND_{k,i,rg,t}}{IT_t} = \frac{A_{k,i,rg} PK_{k,i,rg,t} KS_{k,i,rg,t} e^{\alpha rs_{k,i,rg,t}}}{e^{\alpha R_t} \sum_{kj} \sum_j \sum_{rgj} A_{kj,j,rgj} PK_{kj,j,rgj,t} KS_{kj,j,rgj,t}} \quad [176]$$

$$\frac{PK_{k,i,rg,t} IND_{k,i,rg,t}}{IT_t} = \frac{A_{k,i,rg} PK_{k,i,rg,t} KS_{k,i,rg,t}}{\sum_{kj} \sum_j \sum_{rgj} A_{kj,j,rgj} PK_{kj,j,rgj,t} KS_{kj,j,rgj,t}} e^{\alpha (rs_{k,i,rg,t} - R_t)} \quad [177]$$

Bchir *et al.* interpret R_t as « an opportunity cost (depreciation and risk premium included) of capital [...] » (p. 120). The dynamics of investment tends to equalize rental rates, and the speed of convergence depends on elasticity α .

The balancing variable B_t can now be written

$$B_t = \frac{e^{-\alpha R_t} IT_t}{\sum_{kj} \sum_j \sum_{rgj} A_{kj,j,rgj} PK_{kj,j,rgj,t} KS_{kj,j,rgj,t}} \quad [178]$$

Unfortunately, Bchir *et al.* give only a cursory justification of that formulation. They write (my translation) :

« From a theoretical point of view, the modelling of FDI's [Foreign Direct Investments] in *MIRAGE* must be compatible with that used for national investment, and it must be consistent with a rational behavior on the part of investors in the allocation of their savings. The rental rate of capital is, in that context, a natural determinant of the distribution among industries and countries. On the other hand, that rental rate incorporates the influence of several determinants of FDI's identified in the empirical literature, [...] such as market size, its growth rate, or mercantile potential. So it would be inconsistent to take into account those determinants over and above the industry rental rate of capital. Finally, empirical studies show that the elasticity of investment to the rental rate of capital is finite.

On the basis of these different elements, a single formulation is used to determine both domestic and foreign investment. It proceeds from an allocation of agents' savings between different industries and zones, depending on the initial structure of their savings, the current stock of capital, and the industry rate or return, with an elasticity of α ».

One may conclude that, in the eyes of the authors, the calibrated parameters $A_{k,i,rg}$ represent the « initial structure of their savings », that is, the initial distribution of the stock of capital ; that is quite correct, as we have seen in 2.2. And of course, $KS_{k,i,rg,t}$ is the « current stock of capital ».

Now, if, as we have seen, the model is a variety of the gravity model, it can also be interpreted as a multinomial logit model. In this instance, the systematic utility function that leads to the *MIRAGE* specification is

$$v_{k,i,rg,t} = \ln(A_{k,i,rg} PK_{k,i,rg,t} KS_{k,i,rg,t}) + \alpha rs_{k,i,rg,t} \quad [179]$$

That is easily verified by substituting [179] into [140]. What is interesting in the multinomial logit interpretation of this model is that it underlines the *ad hoc* nature of the first term of the utility function, which constitutes its « inertial » part.

3.4 ABBINK, BRABER AND COHEN (1995)

Abbink, Braber and Cohen (1995), present a recursive CGE model of Indonesia where investment shares are determined according to the ratio of industry profitability to average profitability. Industries where investments are more profitable see their share of total investment increase in the future. Those (endogenous) industry shares are defined as

$$\theta_{it} = \frac{\theta_{i0} \left(\frac{PR_{it}}{APR_t} \right)}{\sum_j \theta_{j0} \left(\frac{PR_{jt}}{APR_t} \right)} \quad [180]$$

with

θ_{i0} : industry investment shares in the reference year;

PR_{it} : industry profit rates;

APR_t : average profit rate.

The initial shares are supposed to be equal to industry shares of capital income :

$$\theta_{i0} = \frac{R_{i0} KD_{i0}}{\sum_j R_{j0} KD_{j0}} \quad [181]$$

where R_{i0} and KD_{i0} represent the rental rate and the stock of capital in the reference year.

Industry profit rates are determined by means of the following expression :

$$PR_{it} = \frac{R_{it} KD_{it} - \delta KD_{it} PK_t}{KD_{it} PK_t} \quad [182]$$

where δ is the rate of depreciation, and PK_t is the replacement price of capital. That equation represents the profitability of past investments, which is different from the marginal return to capital. The numerator is the difference between :

- the gross operating surplus, which is the product of the rental rate by the volume of capital, and
- depreciation.

In the denominator, we have the value of the stock of capital. The average profit rate is computed as the weighed mean of profit rates, with weights equal to industry shares in invested capital :

$$APR_t = \sum_i \left(\frac{KD_{it} PK_t}{\sum_j KD_{jt} PK_t} PR_{it} \right) \quad [183]$$

which, given that the price of capital PK_t is the same in all industries, is strictly equivalent to

$$APR_t = \sum_i \left(\frac{KD_{it}}{\sum_j KD_{jt}} PR_{it} \right) \quad [184]$$

Finally, capital accumulation takes the standard form

$$KD_{i,t+1} = (1 - \delta)KD_{it} + \theta_{it} IT_t \quad [185]$$

where IT_t designates the total *volume* of investment. The second term on the right-hand side of that equation represents investment by destination industry.

3.5 THURLOW (2003), AND DERVIS, DE MELO AND ROBINSON (1982)

Likewise, Thurlow (2003), in his South African model, determines distributive shares of investments by means of the following expression :

$$\eta_{it} = \left(\frac{KD_{it}}{\sum_j KD_{jt}} \right) \left[\beta_i \left(\frac{R_{it}}{RM_t} - 1 \right) + 1 \right] \quad [186]$$

where

η_{it} is the share of investment directed to industry i at time t ,

KD_{it} is the quantity of capital in industry i at time t ,

β_i is a parameter;

R_{it} is the rental rate;

RM_t is the average rental rate, defined as :

$$RM_t = \sum_i \left(\frac{KD_{it}}{\sum_j KD_{jt}} R_{it} \right) \quad [187]$$

That equation is of the same form as the one for APR_t in Abbink *et al.* (1995). But there, PR_{it} denotes the rate of return on invested capital (ratio of capital income to the value of invested capital), while here, R_{it} designates the rental rate of capital (ratio of capital income to the quantity of capital, or income per unit of capital).

The share of investments directed to an industry is greater or smaller than its share of existing capital, depending on whether its rental rate is greater or smaller than the average rate. Parameter β_i conveys the sensitivity of investment to differences in rental rates. That can be seen more clearly by rewriting the equation as

$$\eta_{it} = \left(\frac{KD_{it}}{\sum_j KD_{jt}} \right) \left[\beta_i \left(\frac{R_{it}}{RM_t} \right) + (1 - \beta_i) \right] \quad [188]$$

$$\eta_{it} = (1 - \beta_i) \left(\frac{KD_{it}}{\sum_j KD_{jt}} \right) + \beta_i \left(\frac{R_{it}}{RM_t} \right) \left(\frac{KD_{it}}{\sum_j KD_{jt}} \right) \quad [189]$$

A fraction β_i of investment is distributed according to the ratio of rental rates, while the rest is distributed according to industry shares of existing capital. In the extreme case where β_i is zero (no mobility), industry shares of new investment are constant, equal to industry shares of existing capital.

If parameter β_i is 1, the equation of investment distributive shares amounts to

$$\eta_{it} = \left(\frac{KD_{it}}{\sum_j KD_{jt}} \right) \left(\frac{R_{it}}{RM_t} \right) \quad [190]$$

$$\left[\begin{array}{c} \text{Share of new} \\ \text{investment} \end{array} \right] = \left[\begin{array}{c} \text{Share of} \\ \text{existing capital} \end{array} \right] \cdot \left[\begin{array}{c} \text{Ratio of industry rental rate} \\ \text{to average rental rate} \end{array} \right]$$

The total value of investments is the sum or the products of investment by origin INV_{it} by corresponding composite commodity prices PC_{jt} . The total volume of investment is the ratio of the value of investments over the replacement price of capital. Investment by destination industry, IND_{it} , is then simply equal to the product of industry share by the total volume of investments :

$$IND_{it} = \eta_{it} \left(\frac{\sum_j PC_{jt} INV_{jt}}{PK_t} \right) \quad [191]$$

where

PC_{jt} is the price of commodity j in period t ;

INV_{jt} is the quantity demanded of commodity j for investment purposes;

the sum $\sum_j PC_{jt} INV_{jt}$ is total investment spending;

PK_t is the replacement price of capital in period t .

Price index PK_t is

$$PK_t = \frac{\sum_j PC_{jt} INV_{jt}}{\sum_j INV_{jt}} \quad [192]$$

Finally, the accumulation equation is

$$KD_{i,t+1} = KD_{it} \left(1 + \frac{IND_{it}}{KD_{it}} - \delta \right) \quad [193]$$

or, equivalently,

$$KD_{i,t+1} = (1 - \delta) KD_{it} + IND_{it} \quad [194]$$

Thurlow's (2003) specification found its inspiration in the dynamic investment share equation proposed by Dervis, de Melo and Robinson (1982) :

$$H_{i,t+1} = SP_{it} + \mu SP_{it} \left(\frac{R_{it} - AR_t}{AR_t} \right) \quad [195]$$

where

$$SP_{it} = \frac{R_{it}KD_{it}}{\sum_j R_{jt}KD_{jt}} : \text{industry shares of profits;} \quad [196]$$

μ : investment funds mobility parameter;

R_{it} : industry rental rate;

AR_t : average rental rate.

That equation can be rewritten as

$$H_{i,t+1} = SP_{it} \left[\mu \left(\frac{R_{it}}{AR_t} - 1 \right) + 1 \right] \quad [197]$$

The latter is of the same form as the one used by Thurlow (2003), but here, SP_{it} is the industry share of capital *rent*, rather than of existing capital.

3.6 DUMONT AND MESPLÉ-SOMPS (2000)

The Dumont and Mesplé-Somps (2000) model is totally mechanical : total investment is distributed among industries in fixed shares.

$$K_{i,t+1} = (1 - dep)K_{i,t} + \theta_i IT_t \quad [198]$$

It should be pointed out that the object of that model was rather the specification of total private investment IT_t , in such a way that it makes explicit the effect of public on private investment. But here, we want to examine models of the distribution of investment among industries.

4. Synthesis of the theory and surveyed applications

4.1 OVERVIEW OF AVAILABLE CHOICES

Based on neoclassical investment demand theory (Nickell, 1978), we derived the discrete-time dynamic investment demand model with adjustment costs.

Under the assumption that

- adjustment costs are independent of the stock of capital, of the form

$$C(I_t) = q_t \frac{\gamma}{2} I_t^2 \quad [062]$$

where q_t is the replacement price of capital, that is, the price of the investment good in period t

- expectations are stationary :

$$\tilde{R}_{t+s} = R_t, \forall s \geq 0 \quad [054]$$

$$\text{where } R_t = p_t \frac{\partial F}{\partial K_t} \quad [052]$$

the investment demand function is

$$I_t = \frac{1}{\gamma} \left(\frac{R_t}{\bar{u}_t} - 1 \right) \quad [111]$$

$$\text{where } \bar{u}_t = (r + \delta) q_t \quad [056]$$

That is equivalent to the theoretical investment demand function of Bourguignon *et al.* (1989) :

$$I_t = a \left\{ \frac{p^n MP_k U}{q(\delta + J^F)} - 1 \right\} = a \left\{ \frac{B}{C} - 1 \right\} \geq 0 \quad [112]$$

with the exception that the latter takes into account the capacity utilization rate U .

But these authors turn away from that specification, because it leads to extreme fluctuations in their model. They substitute the *ad hoc* quadratic function

$$\frac{I_t}{K_t} = q\gamma_1 \left[\left(\frac{B}{C} \right)^2 + \gamma_2 \left(\frac{B}{C} \right) \right] \quad [113]$$

We also examined Jung and Thorbecke's (2001) equation for investment by destination industry :

$$\frac{INV_{it}}{K_{it}} = A_i \left(\frac{KINC_{it}}{PK_{it} K_{it} r_t} \right)^{\beta_i} \quad [115]$$

and that of Fargeix and Sadoulet (1994) :

$$\frac{I_{it}}{K_{it}} = B_i \left(\frac{KINC_{it} (1 + \pi_t)}{PK_{it} K_{it} (1 + rd_t)} \right)^{\varepsilon_i} \quad [120]$$

None of these two models takes depreciation into account; but Jung and Thorbecke's model can be modified in order to do so.

We have shown how both of these formulations relate to Tobin's q . In the first case, the investment rate is a constant elasticity function of a version of q . In the second case, the q is replaced by a related entity : the ratio $\frac{KINC_{it}(1+\pi_t)}{1+rd_t}$ is the present value at time t of an income

of $KINC_{it}(1+\pi_t)$ received in period $t+1$, discounted at rate rd_t . The ratio $\left(\frac{KINC_{it}(1+\pi_t)}{PK_{it}K_{it}(1+rd_t)}\right)$ could be called a truncated q , or a single-future-period q .

Agénor's (2003) model is similar to that of Jung and Thorbecke (2001), with the difference that it takes depreciation and inflation into account. That of Collange (1993), which aims to capture the effects of financing constraints (or rather, the importance of self-financing in developing countries) is more of an *ad hoc* nature, by the author's own account.

A model of industry investment demand, together with the hypothesis that supply is perfectly inelastic, independent of capital income, and determined by the savings-investment equality constraint, constitutes a complete specification. We have nonetheless examined other modeling options.

The capital vintage model of Beghin *et al.* (1996) and of Mensbrughe (2003) proposes an altogether different approach. There is no *investment* demand as such in that model. There is only demand and supply of *capital*, with perfect mobility for new capital, and partial mobility for old capital released by declining industries. And the aggregate supply of capital is independent of capital income. For old capital is inherited from the preceding period; as for new capital, it is simply the ratio of the preceding period's savings to the aggregate price of investment in the same period, while foreign savings are exogenous and household savings are a constant fraction of supernumerary income²⁰.

The investment distribution function in the *MIRAGE* model of Bchir *et al.* (2002, p.119), rewritten in our notation, is given by

$$\frac{PK_{k,t}IND_{k,i,rg,t}}{IT_t} = \frac{A_{k,i,rg}PK_{k,t}KS_{k,i,rg,t}e^{\alpha rs_{k,i,rg,t}}}{\sum_{kj} \sum_j \sum_{rgj} A_{kj,j,rgj}PK_{kj,t}KS_{kj,j,rgj,t}e^{\alpha rs_{kj,j,rgj,t}}} \quad [168]$$

²⁰ In Mensbrughe's (2003) simplified version, there is no government, and no business savings. But even in the more elaborate version of Beghin *et al.* (1996), the aggregate supply of capital remains independent of the rental rate of capital.

That corresponds to the gravity model, based upon an analogy with Newton's Law of gravity, where the « force of attraction » of a destination is given by the stock of capital, evaluated at replacement cost $PK_{k,t} KS_{k,i,rg,t}$ and where the « friction of distance » is an inverse function of the rental rate of capital $rs_{k,i,rg,t}$

We have also explored the multinomial logit model, which is based on the concept of random utility. According to that discrete choice model, each individual rationally chooses the possibility which yields the greatest utility for him/her. But the utility of a given possibility for a given individual is not deterministic, because investors are not unanimous in their expectations : since individual expectations depend on a large number of factors, several of which are unobservable, it is suitable to represent those variations by means of a random term. The utility of investment i for investor n can be written as

$$U_{in} = \beta_i v_i + \varepsilon_{in} \quad [138]$$

where

$$v_i = \left(\frac{rs_i}{TIN} - PK_i \right) \text{ is the net present value of an investment of one unit of capital into industry } i$$

under the assumption of stationary expectations;

rs_i is the rental rate of capital in industry i ;

β_i is the parameter describing the sensitivity of investors to the v_i 's;

$\beta_i v_i$ is therefore the systematic part of utility; parameter β_i is necessary to define the relative weight of systematic utility relative to the random term;

ε_{in} is a random term.

It follows that the probability that investor n choose destination industry i is

$$\Pr_n(i) = \frac{\exp(\beta_i v_i)}{\sum_j \exp(\beta_j v_j)} \quad [140]$$

With a large number of investors n , identical except for the value of the random terms, the probability $\Pr_n(i)$ gives the distribution of investments among industries.

The investment equation in the *MIRAGE* model can also be interpreted as a multinomial logit model. In this instance, the systematic utility function that leads to the *MIRAGE* equation is

$$v_{k,i,rg,t} = \ln(A_{k,i,rg} PK_{k,t} KS_{k,i,rg,t}) + \alpha rs_{k,i,rg,t} \quad [179]$$

What is interesting in the multinomial logit interpretation of this model is that it underlines the *ad hoc* nature of the first term of the utility function, which constitutes its « inertial » part.

Among the other models surveyed, most (Abbink, Braber and n, 1995 ; Thurlow, 2003 and Dervis, de Melo and Robinson, 1982) are models of the distribution of investments depending on relative rates of return or rental rates of capital. Although those specifications seem sensible, their theoretical foundations are not formally stated.

4.2 LINKS WITH THE ISSUES OF SAVINGS AND DEBT

With *MIRAGE*'s gravity model, or the multinomial logit supply model, there is no *a priori* link between the distribution of investment among industries and savings or debt. On the other hand, with models of investment demand, there are links. The endogenous rate of interest, which plays the part of a discount rate, can create a link between investment demand and current savings and future debt.

For current savings, it is obvious, insofar as savings depend on the real interest rate. As for debt, one can imagine that some future financial flows be determined by the interest rate at which funds for investment purposes have been borrowed in the current period.

4.3 CONCLUSION

A priori, our preference would go to an investment demand model, and more precisely to the model of Bourguignon *et al.* (1989) in its theoretically exact form ²¹ (rather than in the *ad hoc* form of equation [113]). For demand models are those which rest upon the strongest theoretical foundations, and, among them, the Bourguignon *et al.* (1989) one is the most rigorous. But, because of the instability that resulted, those authors have given it up for the *ad hoc* formulation. We have found the same difficulties when we tried to implement the theoretical form in the *EXTER-Debt* model (Lemelin, 2007).

A careful examination revealed the cause of the instability generated by the theoretical form of Bourguignon *et al.* : it is the extremely high value of the elasticity of the accumulation rate relative to the $\frac{R_t}{\bar{u}_t}$ ratio in [111]. The Jung-Thorbecke form raises the same problem when the

²¹ In which case consistency would require that the adjustment costs underlying that specification be taken into account in the CGE model.

value of the elasticity parameter is set too high. It is therefore infeasible to apply a theoretically exact investment demand function.

For all that, we do not propose to retain the pragmatic solution of Bourguignon *et al.*, because we find it unsatisfactory, for two reasons. First, it is quite restrictive with respect to the elasticity of the accumulation rate relative to r_i/u_i ; in that respect, the Jung-Thorbecke form offers greater flexibility. Second, the calibration procedure of Bourguignon *et al.* imposes

$$\frac{R_0}{\bar{u}_0} = 1 \quad [199]$$

where R_0 and \bar{u}_0 are the initial values of R_t and \bar{u}_t respectively. Yet, according to the theoretical model, that condition is precisely the one that should lead to a null gross rate of accumulation, that is, to a rate of accumulation net of depreciation that would be negative.

In light of the above, we would settle for an investment function similar to that of Jung and Thorbecke (2001) :

$$\frac{INV_{it}}{K_{it}} = A_i \left(\frac{KINC_{it}}{PK_{it} K_{it} (r_t + \delta)} \right)^{\beta_i} \quad [118]$$

where parameter A_i is calibrated to yield a regular path, characterized by

$$\frac{INV_{it}}{K_{it}} = A_i \left(\frac{KINC_{it}}{PK_{it} K_{it} (r_t + \delta)} \right)^{\beta_i} = g + \delta \quad [200]$$

where g is the exogenous growth rate of the supply of labor. Parameter A_i can be calibrated by means of

$$A_i = (g + \delta) \left(\frac{KINC_{it}}{PK_{it} K_{it} (r_t + \delta)} \right)^{-\beta_i} \quad [201]$$

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Appendix A1 : A theoretical model with first-degree homogenous adjustment costs

In his Chapter 3, Nickell (1978) develops a theoretical model with adjustment costs²². He starts with the following hypothesis :

8. There are adjustment costs associated with variations in the capital stock. These costs are a function of gross investment, they increase with the absolute value of investment or disinvestment, and, moreover, they increase at an increasing rate. They are null only when gross investment is null.

Formally, that implies an adjustment cost function $C(I)$ with the following properties (Nickell, 1978, p. 27) :

$$C'(I_t) > 0 \text{ and } C''(I_t) > 0 \Leftrightarrow I_t > 0 \text{ and } I_t < 0 \quad [059]$$

$$C(0) = 0 \quad [060]$$

$$C''(I_t) > 0 \quad [061]$$

Among the functional forms with those properties, there is²³ :

$$C(I_t, K_t) = q_t \frac{\gamma}{2} \frac{I_t^2}{K_t} \quad [082]$$

Adjustment costs are a function of the volume of investment, and inversely proportional to the stock of capital, so the conditions of Hayashi (1982) are fulfilled.

A1.1 FIRST-ORDER OPTIMUM CONDITIONS

As before, the firm maximizes the present value of its cash flow. If one supposes the discount rate to be constant, the maximization problem is

²² Épaulard (1993) calls them installation costs.

²³ That form is presented with some variants in Nabil Annabi, *Les MEGC avec anticipations rationnelles : introduction*, slide presentation, March 2003; slide No. 33 and following. It should be noted that that adjustment cost function is sometimes written as

$$C(I_t, K_t) = \frac{\gamma}{2} \frac{I_t^2}{K_t}$$

in which case [202] becomes

$$MAX V = \sum_{t=0}^{\infty} \frac{1}{(1+r)^t} [p_t F(K_t, L_t) - w_t L_t - q_t I_t - q_t C(I_t, K_t)]$$

$$MAX V = \sum_{t=0}^{\infty} \frac{1}{(1+r)^t} [p_t F(K_t, L_t) - w_t L_t - q_t I_t - C(I_t, K_t)] \quad [202]$$

$$\text{s.c. } I_t = K_{t+1} - K_t + \delta K_t = K_{t+1} - (1-\delta)K_t \quad [014]$$

$$\text{and } K_0 = \bar{K}_0 \quad [005]$$

Substituting the adjustment cost function, the objective function becomes

$$MAX V = \sum_{t=0}^{\infty} \frac{1}{(1+r)^t} \left[p_t F(K_t, L_t) - w_t L_t - q_t I_t - q_t \frac{\gamma}{2} \frac{I_t^2}{K_t} \right] \quad [203]$$

$$MAX V = \sum_{t=0}^{\infty} \frac{1}{(1+r)^t} \left[p_t F(K_t, L_t) - w_t L_t - q_t I_t \left(1 + \frac{\gamma}{2} \frac{I_t}{K_t} \right) \right] \quad [204]$$

Write the Lagrangian

$$\Lambda = \sum_{t=0}^{\infty} \frac{1}{(1+r)^t} \left\{ p_t F(K_t, L_t) - w_t L_t - q_t I_t \left(1 + \frac{\gamma}{2} \frac{I_t}{K_t} \right) + \lambda_t [I_t - K_{t+1} + (1-\delta)K_t] \right\} - \mu (K_0 - \bar{K}_0) \quad [205]$$

The solution leads to first-order conditions

$$\frac{\partial \Lambda}{\partial L_t} = \frac{1}{(1+r)^t} \left[p_t \frac{\partial F}{\partial L_t} - w_t \right] = 0 \quad [206]$$

$$\frac{\partial \Lambda}{\partial I_t} = \frac{1}{(1+r)^t} \left[q_t \left(-1 - \gamma \frac{I_t}{K_t} \right) + \lambda_t \right] = \frac{1}{(1+r)^t} \left[-q_t \left(1 + \gamma \frac{I_t}{K_t} \right) + \lambda_t \right] = 0 \quad [207]$$

$$\frac{\partial \Lambda}{\partial K_t} = \frac{1}{(1+r)^t} \left[p_t \frac{\partial F}{\partial K_t} + q_t \frac{\gamma}{2} \frac{I_t^2}{K_t^2} - (1+r)\lambda_{t-1} + (1-\delta)\lambda_t \right] = 0 \quad [208]$$

$$\frac{\partial \Lambda}{\partial \lambda_t} = \frac{1}{(1+r)^t} [I_t - K_{t+1} + (1-\delta)K_t] \quad [209]$$

$$\frac{\partial \Lambda}{\partial \mu} = -(K_0 - \bar{K}_0) = 0 \quad [210]$$

Condition [207] is equivalent to

$$\lambda_t = q_t \left(1 + \gamma \frac{I_t}{K_t} \right) \quad [211]$$

Substituting for λ_{t-1} and λ_t in [208],

$$\frac{\partial \Lambda}{\partial K_t} = \frac{1}{(1+r)^t} \left[p_t \frac{\partial F}{\partial K_t} + q_t \frac{\gamma}{2} \frac{I_t^2}{K_t^2} - (1+r) q_{t-1} \left(1 + \gamma \frac{I_{t-1}}{K_{t-1}} \right) + (1-\delta) q_t \left(1 + \gamma \frac{I_t}{K_t} \right) \right] = 0 \quad [212]$$

$$p_t \frac{\partial F}{\partial K_t} + q_t \frac{\gamma}{2} \frac{I_t^2}{K_t^2} - (1+r) q_{t-1} \left(1 + \gamma \frac{I_{t-1}}{K_{t-1}} \right) + (1-\delta) q_t \left(1 + \gamma \frac{I_t}{K_t} \right) = 0 \quad [213]$$

First-order conditions become

$$p_t \frac{\partial F}{\partial L_t} = w \quad [016]$$

$$p_t \frac{\partial F}{\partial K_t} + q_t \frac{\gamma}{2} \frac{I_t^2}{K_t^2} - (1+r) q_{t-1} \left(1 + \gamma \frac{I_{t-1}}{K_{t-1}} \right) + (1-\delta) q_t \left(1 + \gamma \frac{I_t}{K_t} \right) = 0 \quad [213]$$

$$I_t = K_{t+1} - (1-\delta) K_t \quad [014]$$

$$K_0 = \bar{K}_0 \quad [005]$$

Let

$$Q_t = \frac{\partial}{\partial I_t} \left[q_t I_t \left(1 + \gamma \frac{I_t}{K_t} \right) \right] = q_t \left(1 + \gamma \frac{I_t}{K_t} \right) \quad [214]$$

It is the marginal cost, or implicit replacement price of investment with adjustment costs.

Equation [213] can be rewritten

$$p_t \frac{\partial F}{\partial K_t} + q_t \frac{\gamma}{2} \frac{I_t^2}{K_t^2} = (1+r) Q_{t-1} - (1-\delta) Q_t \quad [215]$$

A1.2 USER COST OF CAPITAL WITH ADJUSTMENT COSTS

Define the retrospective rate of increase of the marginal cost of capital as

$$\Pi_t = \frac{(Q_t - Q_{t-1})}{Q_{t-1}} \quad [067]$$

so that

$$(Q_t - Q_{t-1}) = \frac{(Q_t - Q_{t-1})}{Q_{t-1}} Q_{t-1} = \Pi_t Q_{t-1} \quad [216]$$

We can rewrite [213] and [215]

$$\left(p_t \frac{\partial F}{\partial K_t} + q_t \frac{\gamma}{2} \frac{I_t^2}{K_t^2} \right) = r Q_{t-1} + \delta Q_t - (Q_t - Q_{t-1}) \quad [217]$$

$$\left(p_t \frac{\partial F}{\partial K_t} + q_t \frac{\gamma}{2} \frac{I_t^2}{K_t^2} \right) = (r - \Pi_t) Q_{t-1} + \delta Q_t \quad [218]$$

where

$$q_t \frac{\gamma}{2} \frac{I_t^2}{K_t^2} = - \frac{\partial}{\partial K_t} \left(q_t \frac{\gamma}{2} \frac{I_t^2}{K_t^2} \right) = - \frac{\partial C(I_t, K_t)}{\partial K_t} \quad [219]$$

is the marginal value of adjustment costs avoided in period t . The right-hand side of [218] is therefore the sum of the value of the marginal product of capital in period t , and of the marginal value of adjustment costs avoided in period t . So let

$$\frac{\partial \Phi_t}{\partial K_t} = p_t \frac{\partial F}{\partial K_t} - \frac{\partial C(I_t, K_t)}{\partial K_t} = \left(p_t \frac{\partial F}{\partial K_t} + q_t \frac{\gamma}{2} \frac{I_t^2}{K_t^2} \right) \quad [220]$$

On the other hand, the left-hand side of [218] is the user cost of capital with adjustment costs :

$$U_t = (r - \Pi_t) Q_{t-1} + \delta Q_t \quad [071]$$

Condition [218] can be written

$$\frac{\partial \Phi_t}{\partial K_t} = (r - \Pi_t) Q_{t-1} + \delta Q_t = U_t \quad [221]$$

It is equivalent to [021], modified to take account of adjustment costs.

A1.3 TOBIN'S q IN THE FIRST-ORDER CONDITIONS

Where can Tobin's q be found in the model stated above?

Recall that

$$\frac{\partial \Phi_t}{\partial K_t} = p_t \frac{\partial F}{\partial K_t} - \frac{\partial C(I_t, K_t)}{\partial K_t} = \left(p_t \frac{\partial F}{\partial K_t} + q_t \frac{\gamma}{2} \frac{I_t^2}{K_t^2} \right) \quad [220]$$

is the sum of the value of the marginal product of capital in period t and of the marginal value of adjustment costs avoided in period t .

Develop²⁴

$$p_t \frac{\partial F}{\partial K_t} + q_t \frac{\gamma}{2} \frac{l_t^2}{K_t^2} = (1+r)Q_{t-1} - (1-\delta)Q_t \quad [215]$$

$$\frac{\partial \Phi_t}{\partial K_t} = (1+r)Q_{t-1} - (1-\delta)Q_t \quad [222]$$

$$(1+r)Q_{t-1} = (1-\delta)Q_t + \frac{\partial \Phi_t}{\partial K_t} \quad [223]$$

$$(1+r)Q_t = (1-\delta)Q_{t+1} + \frac{\partial \Phi_{t+1}}{\partial K_{t+1}} \quad [224]$$

$$Q_t = \frac{1}{(1+r)} \left[(1-\delta)Q_{t+1} + \frac{\partial \Phi_{t+1}}{\partial K_{t+1}} \right] \quad [225]$$

$$Q_t K_{t+1} = \frac{1}{(1+r)} \left[(1-\delta)Q_{t+1} K_{t+1} + \frac{\partial \Phi_{t+1}}{\partial K_{t+1}} K_{t+1} \right] \quad [226]$$

Now, the accumulation constraint

$$l_t = K_{t+1} - (1-\delta)K_t \quad [014]$$

amounts to

$$(1-\delta)K_t = K_{t+1} - l_t \quad [030]$$

Substituting, there results

$$Q_t K_{t+1} = \frac{1}{(1+r)} \left(Q_{t+1} (K_{t+2} - l_{t+1}) + \frac{\partial \Phi_{t+1}}{\partial K_{t+1}} K_{t+1} \right) \quad [227]$$

$$Q_t K_{t+1} = \frac{1}{(1+r)} \left(Q_{t+1} K_{t+2} - Q_{t+1} l_{t+1} + \frac{\partial \Phi_{t+1}}{\partial K_{t+1}} K_{t+1} \right) \quad [228]$$

$$Q_t K_{t+1} = \frac{1}{(1+r)} \left(\frac{\partial \Phi_{t+1}}{\partial K_{t+1}} K_{t+1} - Q_{t+1} l_{t+1} + Q_{t+1} K_{t+2} \right) \quad [229]$$

where the term $Q_{t+1} K_{t+2}$ can be replaced by its expression according to that very same relation.

Then,

²⁴ The development that follows is parallel to Hayashi's (1982), as reproduced in Nabil Annabi, *Les MEGC avec anticipations rationnelles : introduction*, slide presentation, March 2003; see slides No. 38 and following.

$$Q_t K_{t+1} = \frac{1}{(1+r)} \left[\frac{\partial \Phi_{t+1}}{\partial K_{t+1}} K_{t+1} - Q_{t+1} I_{t+1} \right. \\ \left. + \frac{1}{(1+r)} \left(\frac{\partial \Phi_{t+2}}{\partial K_{t+2}} K_{t+2} - Q_{t+2} I_{t+2} + Q_{t+2} K_{t+3} \right) \right] \quad [230]$$

Successive substitutions lead to

$$Q_t K_{t+1} = \sum_{s=1}^{\infty} \frac{1}{(1+r)^s} \left[\frac{\partial \Phi_{t+s}}{\partial K_{t+s}} K_{t+s} - Q_{t+s} I_{t+s} \right] + \lim_{s \rightarrow \infty} \left(\frac{1}{(1+r)^s} Q_{t+s} K_{t+s+1} \right) \quad [231]$$

where the last term is null by virtue of transversality condition

$$\lim_{s \rightarrow \infty} \left(\frac{1}{(1+r)^s} Q_{t+s} K_{t+s+1} \right) = 0 \quad [077]$$

Therefore,

$$Q_t K_{t+1} = \sum_{s=1}^{\infty} \frac{1}{(1+r)^s} \left[\frac{\partial \Phi_{t+s}}{\partial K_{t+s}} K_{t+s} - Q_{t+s} I_{t+s} \right] \quad [232]$$

We now depart from Nickell's (1978) hypotheses, and, instead of strictly decreasing returns to scale, we suppose constant returns. The production function $F(K_t, L_t)$ is then first-degree homogenous, which implies Euler's condition

$$F(K_t, L_t) = \frac{\partial F}{\partial K_t} K_t + \frac{\partial F}{\partial L_t} L_t \quad [040]$$

and, equivalently,

$$\frac{\partial F}{\partial K_t} K_t = F(K_t, L_t) - \frac{\partial F}{\partial L_t} L_t \quad [233]$$

$$\rho_t \frac{\partial F}{\partial K_t} K_t = \rho_t F(K_t, L_t) - \rho_t \frac{\partial F}{\partial L_t} L_t \quad [234]$$

Given first-order condition

$$\rho_t \frac{\partial F}{\partial L_t} = w_t \quad [016]$$

Euler's condition reduces to

$$\rho_t \frac{\partial F}{\partial K_t} K_t = \rho_t F(K_t, L_t) - w_t L_t \quad [041]$$

Given

$$\frac{\partial \Phi_t}{\partial K_t} = \left(p_t \frac{\partial F}{\partial K_t} + q_t \frac{\gamma}{2} \frac{I_t^2}{K_t^2} \right) \quad [220]$$

there follows

$$\frac{\partial \Phi_t}{\partial K_t} K_t = \left(p_t F(K_t, L_t) - w_t L_t + q_t \frac{\gamma}{2} \frac{I_t^2}{K_t^2} K_t \right) = \left(p_t F(K_t, L_t) - w_t L_t + q_t \frac{\gamma}{2} \frac{I_t^2}{K_t} \right) \quad [235]$$

Now, replace $\frac{\partial \Phi_{t+s}}{\partial K_{t+s}} K_{t+s}$ in [232] :

$$Q_t K_{t+1} = \sum_{s=1}^{\infty} \frac{1}{(1+r)^s} \left[p_{t+s} F(K_{t+s}, L_{t+s}) - w_{t+s} L_{t+s} + q_{t+s} \frac{\gamma}{2} \frac{I_{t+s}^2}{K_{t+s}} - Q_{t+s} I_{t+s} \right] \quad [236]$$

Also recall that

$$Q_t = \frac{\partial}{\partial I_t} \left[q_t I_t \left(1 + \frac{\gamma}{2} \frac{I_t}{K_t} \right) \right] = q_t \left(1 + \gamma \frac{I_t}{K_t} \right) \quad [214]$$

and let us make [232] more explicit

$$Q_t K_{t+1} = \sum_{s=1}^{\infty} \frac{1}{(1+r)^s} \left[p_{t+s} F(K_{t+s}, L_{t+s}) - w_{t+s} L_{t+s} + q_{t+s} \frac{\gamma}{2} \frac{I_{t+s}^2}{K_{t+s}} - q_{t+s} \left(1 + \gamma \frac{I_{t+s}}{K_{t+s}} \right) I_{t+s} \right] \quad [237]$$

$$Q_t K_{t+1} = \sum_{s=1}^{\infty} \frac{1}{(1+r)^s} \left\{ p_{t+s} F(K_{t+s}, L_{t+s}) - w_{t+s} L_{t+s} + q_{t+s} I_{t+s} \left[\frac{\gamma}{2} \frac{I_{t+s}}{K_{t+s}} - \left(1 + \gamma \frac{I_{t+s}}{K_{t+s}} \right) \right] \right\} \quad [238]$$

$$Q_t K_{t+1} = \sum_{s=1}^{\infty} \frac{1}{(1+r)^s} \left[p_{t+s} F(K_{t+s}, L_{t+s}) - w_{t+s} L_{t+s} - q_{t+s} I_{t+s} \left(1 + \frac{\gamma}{2} \frac{I_{t+s}}{K_{t+s}} \right) \right] \quad [239]$$

that is,

$$\frac{\sum_{s=1}^{\infty} \frac{1}{(1+r)^s} \left[p_{t+s} F(K_{t+s}, L_{t+s}) - w_{t+s} L_{t+s} - q_{t+s} I_{t+s} \left(1 + \frac{\gamma}{2} \frac{I_{t+s}}{K_{t+s}} \right) \right]}{Q_t K_{t+1}} = 1 \quad [083]$$

The numerator of [083] is the present value in period t of the firm's cash flow from period $t+1$ onwards; the discount rate is the market rate, so that that present value corresponds to the stock market valuation of the numerator of Tobin's q . Notice the one-period delay : capital

available in period $t+1$ must have been invested in period t (or re-invested, that is, not dis-invested); so the cash flows to be taken into account are those from period $t+1$. The denominator of [083] is the marginal replacement cost in period t of capital that will be used from period $t+1$ onwards. Note it is the marginal replacement cost, and not the user cost. The left-hand side ratio of [083] is therefore analogous to Tobin's q : investment made in t is optimal when that ratio is equal to 1. But, contrary to Tobin's q , the denominator of the ratio in [083] is not a constant price, but a marginal cost that takes into account adjustment costs.

A1.4 THE INTERTEMPORAL EQUILIBRIUM OF CAPITAL

The condition

$$p_t \frac{\partial F}{\partial K_t} + q_t \frac{\gamma}{2} \frac{I_t^2}{K_t^2} = (1+r)Q_{t-1} - (1-\delta)Q_t \quad [215]$$

is equivalent to

$$(1-\delta)Q_t = (1+r)Q_{t-1} - \left(p_t \frac{\partial F}{\partial K_t} + q_t \frac{\gamma}{2} \frac{I_t^2}{K_t^2} \right) \quad [240]$$

$$Q_t = \frac{(1+r)}{(1-\delta)} Q_{t-1} - \frac{1}{(1-\delta)} \left(p_t \frac{\partial F}{\partial K_t} + q_t \frac{\gamma}{2} \frac{I_t^2}{K_t^2} \right) \quad [241]$$

Optimum is achieved when investment in period t is at the point where its marginal cost Q_t (which grows with I_t) satisfies condition [241].

Advancing one period forward yields

$$Q_{t+1} = \frac{(1+r)}{(1-\delta)} Q_t - \frac{1}{(1-\delta)} \left(p_{t+1} \frac{\partial F}{\partial K_{t+1}} + q_{t+1} \frac{\gamma}{2} \frac{I_{t+1}^2}{K_{t+1}^2} \right) \quad [242]$$

which can be rewritten as

$$Q_t = \frac{1}{(1+r)} \left\{ (1-\delta) Q_{t+1} + \left(p_{t+1} \frac{\partial F}{\partial K_{t+1}} + q_{t+1} \frac{\gamma}{2} \frac{I_{t+1}^2}{K_{t+1}^2} \right) \right\} \quad [243]$$

Advancing that equation to $t+1$, $t+2$, etc., there obtains

$$Q_{t+1} = \frac{1}{(1+r)} \left\{ (1-\delta) Q_{t+2} + \left(p_{t+2} \frac{\partial F}{\partial K_{t+2}} + q_{t+1} \frac{\gamma}{2} \frac{I_{t+2}^2}{K_{t+2}^2} \right) \right\} \quad [244]$$

$$Q_{t+2} = \frac{1}{(1+r)} \left\{ (1-\delta) Q_{t+3} + \left(p_{t+3} \frac{\partial F}{\partial K_{t+3}} + q_{t+1} \frac{\gamma}{2} \frac{I_{t+3}^2}{K_{t+3}^2} \right) \right\} \quad [245]$$

Then, by successive substitutions, we get

$$Q_t = \frac{1}{(1+r)} \left\{ \begin{aligned} & \frac{(1-\delta)^3}{(1+r)^2} Q_{t+3} + \frac{(1-\delta)^2}{(1+r)^2} \left(p_{t+3} \frac{\partial F}{\partial K_{t+3}} + q_{t+3} \frac{\gamma}{2} \frac{I_{t+3}^2}{K_{t+3}^2} \right) \\ & + \frac{(1-\delta)}{(1+r)} \left(p_{t+2} \frac{\partial F}{\partial K_{t+2}} + q_{t+2} \frac{\gamma}{2} \frac{I_{t+2}^2}{K_{t+2}^2} \right) \\ & + \left(p_{t+1} \frac{\partial F}{\partial K_{t+1}} + q_{t+1} \frac{\gamma}{2} \frac{I_{t+1}^2}{K_{t+1}^2} \right) \end{aligned} \right\} \quad [246]$$

or again, after developing,

$$Q_t = \frac{1}{(1+r)} \left\{ \begin{aligned} & \frac{(1-\delta)^s}{(1+r)^{s-1}} Q_{t+s} + \frac{(1-\delta)^{s-1}}{(1+r)^{s-1}} \left(p_{t+s} \frac{\partial F}{\partial K_{t+s}} + q_{t+s} \frac{\gamma}{2} \frac{I_{t+s}^2}{K_{t+s}^2} \right) \\ & \vdots \\ & + \frac{(1-\delta)^2}{(1+r)^2} \left(p_{t+3} \frac{\partial F}{\partial K_{t+3}} + q_{t+3} \frac{\gamma}{2} \frac{I_{t+3}^2}{K_{t+3}^2} \right) \\ & + \frac{(1-\delta)}{(1+r)} \left(p_{t+2} \frac{\partial F}{\partial K_{t+2}} + q_{t+2} \frac{\gamma}{2} \frac{I_{t+2}^2}{K_{t+2}^2} \right) \\ & + \left(p_{t+1} \frac{\partial F}{\partial K_{t+1}} + q_{t+1} \frac{\gamma}{2} \frac{I_{t+1}^2}{K_{t+1}^2} \right) \end{aligned} \right\} \quad [247]$$

The no-speculative-bubbles condition,

$$\lim_{t \rightarrow \infty} \left(\frac{1-\delta}{1+r} \right)^t Q_t = 0 \quad [087]$$

combined with [247], results in

$$Q_t = \frac{1}{(1+r)} \sum_{s=1}^{\infty} \left(\frac{1-\delta}{1+r} \right)^{s-1} \left(p_{t+s} \frac{\partial F}{\partial K_{t+s}} + q_{t+s} \frac{\gamma}{2} \frac{I_{t+s}^2}{K_{t+s}^2} \right) \quad [248]$$

Denote the value of the marginal productivity of capital

$$R_t = p_t \frac{\partial F}{\partial K_t} \quad [052]$$

and, as before, the adjustment cost function

$$C(I_t, K_t) = q_t \frac{\gamma}{2} \frac{I_t^2}{K_t} \quad [082]$$

and

$$\frac{\partial C(I_t, K_t)}{\partial K_t} = -q_t \frac{\gamma}{2} \frac{I_t^2}{K_t^2} = -\frac{C(I_t, K_t)}{K_t} \quad [249]$$

We can write

$$Q_t = \frac{1}{(1+r)} \sum_{s=1}^{\infty} \left(\frac{1-\delta}{1+r} \right)^{s-1} \left(R_{t+s} - \frac{\partial C(I_{t+s}, K_{t+s})}{\partial K_{t+s}} \right) \quad [250]$$

or, equivalently,

$$Q_t = \frac{1}{(1+r)} \sum_{s=0}^{\infty} \left(\frac{1-\delta}{1+r} \right)^s \left(R_{t+s+1} - \frac{\partial C(I_{t+s+1}, K_{t+s+1})}{\partial K_{t+s+1}} \right) \quad [251]$$

$$Q_t = \frac{1}{(1+r)} \sum_{s=0}^{\infty} \left(\frac{1-\delta}{1+r} \right)^s \left(R_{t+s+1} + \frac{C(I_{t+s+1}, K_{t+s+1})}{K_{t+s+1}} \right) \quad [252]$$

Recall that Q_t is the marginal cost of new capital

$$Q_t = \frac{\partial}{\partial I_t} \left[q_t I_t \left(1 + \frac{\gamma}{2} \frac{I_t}{K_t} \right) \right] = q_t \left(1 + \gamma \frac{I_t}{K_t} \right) \quad [214]$$

Condition [252] shows that, at optimum, the marginal cost of new capital must be equal to its marginal revenue, which is the discounted sum of the future income flows it generates; those future incomes consist of (1) the values R_{t+s} of marginal products, and (2) avoided adjustment costs, given by

$$-\frac{\partial C(I_{t+s}, K_{t+s})}{\partial K_{t+s}} > 0$$

These flows decrease with time, as capital depreciates; whence attrition factor $(1-\delta)^{s-1}$.

A1.5 INVESTMENT DEMAND WITH STATIONARY EXPECTATIONS

The equation for the marginal cost of new capital

$$Q_t = \frac{\partial}{\partial I_t} \left[q_t I_t \left(1 + \frac{\gamma}{2} \frac{I_t}{K_t} \right) \right] = q_t \left(1 + \gamma \frac{I_t}{K_t} \right) \quad [214]$$

amounts to

$$\frac{I_t}{K_t} = \frac{1}{\gamma} \left(\frac{Q_t}{q_t} - 1 \right) \quad [253]$$

But the equation

$$Q_t = \frac{1}{(1+r)} \sum_{s=0}^{\infty} \left(\frac{1-\delta}{1+r} \right)^s \left(R_{t+s+1} + \frac{C(I_{t+s+1}, K_{t+s+1})}{K_{t+s+1}} \right) \quad [252]$$

shows that Q_t depends on future values of R_{t+s} , so that its value is unknown in period t , unless assumptions are made concerning expected values of R_{t+s} . Suppose that \tilde{R}_{t+s} , the expected value at time t of R_{t+s} , is constant (stationary expectations) :

$$\tilde{R}_{t+s} = R_t, \forall s \geq 0 \quad [054]$$

We have already examined the issue of consistency between that hypothesis and the model in 1.2.6. Let us also assume that the expected value of

$$\frac{C(I_{t+s}, K_{t+s})}{K_{t+s}} = q_{t+s} \frac{\gamma}{2} \frac{I_{t+s}^2}{K_{t+s}^2} \quad [254]$$

is constant too, equal to

$$\frac{C(I_t, K_t)}{K_t} = q_t \frac{\gamma}{2} \frac{I_t^2}{K_t^2} \quad [255]$$

Is that second hypothesis sensible? It could be derived from two other hypotheses :

- stationary expectations relative to the price of the investment good q_t :

$$\tilde{q}_{t+s} = q_t$$

- constant expected rate of accumulation, equal to the current rate : $\tilde{g} = \frac{I_t}{K_t}$; it should be

noted that, at this stage, the value of $\tilde{g} = \frac{I_t}{K_t}$ is still unknown.

It then follows that

$$Q_t = \frac{1}{(1+r)} \sum_{s=0}^{\infty} \left(\frac{1-\delta}{1+r} \right)^s \left(\tilde{R}_{t+s+1} + \tilde{q}_{t+s+1} \frac{\gamma}{2} \tilde{g}^2 \right) \quad [252]$$

$$Q_t = \frac{1}{(1+r)} \sum_{s=0}^{\infty} \left(\frac{1-\delta}{1+r} \right)^s \left(R_t + q_t \frac{\gamma}{2} \tilde{g}^2 \right) \quad [256]$$

From the geometric series formula,

$$Q_t = \frac{1}{(1+r)} \left(\frac{1}{1 - \frac{1-\delta}{1+r}} \right) \left(R_t + q_t \frac{\gamma}{2} \tilde{g}^2 \right) \quad [257]$$

$$Q_t = \frac{1}{(1+r)} \frac{1}{\left(\frac{r+\delta}{1+r} \right)} \left(R_t + q_t \frac{\gamma}{2} \tilde{g}^2 \right) \quad [258]$$

$$Q_t = \frac{1}{(r+\delta)} \left(R_t + q_t \frac{\gamma}{2} \tilde{g}^2 \right) \quad [259]$$

Substituting [259] into [253] produces the investment demand equation with stationary expectations in the presence of adjustment costs of form [082] :

$$\frac{I_t}{K_t} = \frac{1}{\gamma} \left[\frac{1}{q_t(r+\delta)} \left(R_t + q_t \frac{\gamma}{2} \tilde{g}^2 \right) - 1 \right] \quad [260]$$

And, given

$$\tilde{g} = \frac{I_t}{K_t} \quad [261]$$

we have

$$\tilde{g} = \frac{1}{\gamma} \left[\frac{1}{q_t(r+\delta)} \left(R_t + q_t \frac{\gamma}{2} \tilde{g}^2 \right) - 1 \right] \quad [262]$$

To find the investment function, it suffices to solve the quadratic equation :

$$(1 + \gamma \tilde{g}) = \frac{1}{q_t(r + \delta)} \left(R_t + q_t \frac{\gamma}{2} \tilde{g}^2 \right) \quad [263]$$

$$(1 + \gamma \tilde{g}) = \frac{R_t}{q_t(r + \delta)} + \frac{1}{q_t(r + \delta)} q_t \frac{\gamma}{2} \tilde{g}^2 \quad [264]$$

$$(1 + \gamma \tilde{g}) = \frac{R_t}{q_t(r + \delta)} + \frac{1}{(r + \delta)} \frac{\gamma}{2} \tilde{g}^2 \quad [265]$$

$$\frac{1}{(r + \delta)} \frac{\gamma}{2} \tilde{g}^2 - \gamma \tilde{g} + \left(\frac{R_t}{q_t(r + \delta)} - 1 \right) = 0 \quad [266]$$

Let

$$A = \frac{1}{(r + \delta)} \frac{\gamma}{2} \quad [267]$$

$$B = -\gamma \quad [268]$$

$$C = \frac{R_t}{q_t(r + \delta)} - 1 \quad [269]$$

and we have

$$\frac{I_t}{K_t} = \tilde{g} = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A} \quad [270]$$

$$\frac{I_t}{K_t} = \tilde{g} = \frac{\gamma \pm \sqrt{\gamma^2 - 4 \frac{1}{(r + \delta)} \frac{\gamma}{2} \left(\frac{R_t}{q_t(r + \delta)} - 1 \right)}}{2 \frac{1}{(r + \delta)} \frac{\gamma}{2}} \quad [271]$$

After simplifying, dividing the numerator and denominator by γ , the investment function obtains

$$\frac{I_t}{K_t} = \tilde{g} = \frac{1 \pm \sqrt{1 - \frac{2}{\gamma} \frac{1}{(r + \delta)} \left(\frac{R_t}{q_t(r + \delta)} - 1 \right)}}{\frac{1}{(r + \delta)}} \quad [272]$$

$$\frac{I_t}{K_t} = \tilde{g} = (r + \delta) \left[1 \pm \sqrt{1 - \frac{2}{\gamma} \frac{1}{(r + \delta)} \left(\frac{R_t}{q_t(r + \delta)} - 1 \right)} \right] \quad [273]$$

The reader will notice that the denominator of R_t in the preceding equation is equal to the user cost of capital with stationary expectations when there are no adjustment costs

$$\bar{u}_t = (r + \delta) q_t \quad [056]$$

However, nothing of what we found in the literature resembles demand function [273].

Annexe A2 : Mathematical developments

A2.1 BASIC DISCRETE-TIME MODEL : FIRST ORDER OPTIMUM CONDITIONS

The firm maximizes the present value of its cash flow. If we assume discount rate r to be constant, and replacing l_t in the objective function by the right-hand side of [014], the maximization problem is

$$MAX V = \sum_{t=0}^{\infty} \frac{1}{(1+r)^t} [p_t F(K_t, L_t) - w_t L_t - q_t l_t] \quad [015]$$

$$\text{s.c. } l_t = K_{t+1} - K_t + \delta K_t = K_{t+1} - (1-\delta)K_t \quad [014]$$

$$\text{and } K_0 = \bar{K}_0 \quad [005]$$

The Lagrangian is

$$\Lambda = \sum_{t=0}^{\infty} \frac{1}{(1+r)^t} \{ [p_t F(K_t, L_t) - w_t L_t - q_t l_t] + \lambda_t [l_t - K_{t+1} + (1-\delta)K_t] \} + \mu (K_0 - \bar{K}_0) \quad [274]$$

Solving leads to the first-order conditions

$$\frac{\partial \Lambda}{\partial L_t} = \frac{1}{(1+r)^t} \left[p_t \frac{\partial F}{\partial L_t} - w_t \right] = 0 \quad [206]$$

$$\frac{\partial \Lambda}{\partial l_t} = \frac{1}{(1+r)^t} (-q_t + \lambda_t) = 0 \quad [275]$$

$$\frac{\partial \Lambda}{\partial K_t} = \frac{1}{(1+r)^t} \left[p_t \frac{\partial F}{\partial K_t} - (1+r)\lambda_{t-1} + \lambda_t(1-\delta) \right] = 0 \quad [276]$$

$$\frac{\partial \Lambda}{\partial \lambda_t} = [l_t - K_{t+1} + (1-\delta)K_t] = 0 \quad [209]$$

$$\frac{\partial \Lambda}{\partial \mu} = (K_0 - \bar{K}_0) = 0 \quad [210]$$

Condition [275] can be written

$$\lambda_t = q_t \quad [277]$$

and the other conditions become

$$p_t \frac{\partial F}{\partial L_t} = w \quad [016]$$

$$p_t \frac{\partial F}{\partial K_t} = (1+r)q_{t-1} - q_t(1-\delta) \quad [017]$$

$$I_t = K_{t+1} - K_t + \delta K_t = K_{t+1} - (1-\delta)K_t \quad [014]$$

$$K_0 = \bar{K}_0 \quad [005]$$

Equation [017] can take the form

$$p_t \frac{\partial F}{\partial K_t} = r q_{t-1} + \delta q_t - (q_t - q_{t-1}) \quad [018]$$

The program defined by conditions [016], [017] and [005] could as well be defined separately for each period. Denote the user cost of capital

$$u_t = (r - \pi_t) q_{t-1} + \delta q_t \quad [022]$$

and state the problem

$$\text{MAX} [p_t F(K_t, L_t) - w_t L_t - u_t K_t] \quad [278]$$

$$\text{s.c. } I_t = K_{t+1} - K_t + \delta K_t = K_{t+1} - (1-\delta)K_t \quad [014]$$

$$\text{and } K_0 = \bar{K}_0 \quad [005]$$

The first-order conditions are

$$p_t \frac{\partial F}{\partial L_t} = w \quad [016]$$

$$p_t \frac{\partial F}{\partial K_t} = u_t \quad [279]$$

$$I_t = K_{t+1} - (1-\delta)K_t \quad [014]$$

$$K_0 = \bar{K}_0 \quad [005]$$

It is easy to verify that, given [021] and [022], conditions [016], [279], [014] and [005] are strictly equivalent to [016], [017], [014] and [005]. However, contrary to continuous-time condition

$$I_t = \dot{K}_t + \delta K_t \quad [003]$$

condition [014] does not involve only current values : the variable K_{t+1} is also present. Although it is possible to state the problem separately for each period, the optimal value of I_t in period t depends of the optimal value of K_{t+1} . The transition from continuous to discrete time implies the

substitution of period-to-period increments to instantaneous rates of increase, so that the intertemporal character is irreducible²⁵.

A2.2 BASIC DISCRETE-TIME MODEL : TOBIN'S q

Develop equation [017]²⁶ :

$$p_t \frac{\partial F}{\partial K_t} = (1+r)q_{t-1} - (1-\delta)q_t \quad [017]$$

$$(1+r)q_{t-1} = (1-\delta)q_t + p_t \frac{\partial F}{\partial K_t} \quad [280]$$

$$(1+r)q_t = (1-\delta)q_{t+1} + p_{t+1} \frac{\partial F}{\partial K_{t+1}} \quad [281]$$

$$q_t = \frac{1}{(1+r)} \left[(1-\delta)q_{t+1} + p_{t+1} \frac{\partial F}{\partial K_{t+1}} \right] \quad [282]$$

$$q_t K_{t+1} = \frac{1}{(1+r)} \left[(1-\delta)q_{t+1} K_{t+1} + p_{t+1} \frac{\partial F}{\partial K_{t+1}} K_{t+1} \right] \quad [029]$$

Now, the accumulation constraint

$$I_t = K_{t+1} - K_t + \delta K_t = K_{t+1} - (1-\delta)K_t \quad [014]$$

amounts to

$$(1-\delta)K_t = K_{t+1} - I_t \quad [030]$$

Substituting into [029] yields

$$q_t K_{t+1} = \frac{1}{(1+r)} \left(q_{t+1} (K_{t+2} - I_{t+1}) + p_{t+1} \frac{\partial F}{\partial K_{t+1}} K_{t+1} \right) \quad [283]$$

$$q_t K_{t+1} = \frac{1}{(1+r)} \left(q_{t+1} K_{t+2} - q_{t+1} I_{t+1} + p_{t+1} \frac{\partial F}{\partial K_{t+1}} K_{t+1} \right) \quad [284]$$

$$q_t K_{t+1} = \frac{1}{(1+r)} \left(p_{t+1} \frac{\partial F}{\partial K_{t+1}} K_{t+1} - q_{t+1} I_{t+1} + q_{t+1} K_{t+2} \right) \quad [031]$$

where the term $q_{t+1} K_{t+2}$ can be replaced by its expression according to that very same relation.

There follows

²⁵ Unless, as in *MIRAGE* (Bchir *et al.*, 2002), currently invested capital is instantaneously productive.

²⁶ The development that follows is parallel to Hayashi's (1982), as reproduced in Nabil Annabi, *Les MEGC avec anticipations rationnelles : introduction*, slide presentation, March 2003; see slides No. 38 and following.

$$q_t K_{t+1} = \frac{1}{(1+r)} \left[p_{t+1} \frac{\partial F}{\partial K_{t+1}} K_{t+1} - q_{t+1} l_{t+1} + \frac{1}{(1+r)} \left(p_{t+2} \frac{\partial F}{\partial K_{t+2}} K_{t+2} - q_{t+2} l_{t+2} + q_{t+2} K_{t+3} \right) \right] \quad [285]$$

Successive substitutions lead to

$$q_t K_{t+1} = \sum_{s=1}^{\infty} \frac{1}{(1+r)^s} \left[p_{t+s} \frac{\partial F}{\partial K_{t+s}} K_{t+s} - q_{t+s} l_{t+s} \right] + \lim_{s \rightarrow \infty} \left(\frac{1}{(1+r)^s} q_{t+s} K_{t+s+1} \right) \quad [032]$$

where the last term is null by virtue of the transversality condition (see inset in section 1.1.3).

$$\lim_{s \rightarrow \infty} \frac{1}{(1+r)^s} q_{t+s} K_{t+s+1} = 0 \quad [033]$$

Therefore,

$$q_t K_{t+1} = \sum_{s=1}^{\infty} \frac{1}{(1+r)^s} \left[p_{t+s} \frac{\partial F}{\partial K_{t+s}} K_{t+s} - q_{t+s} l_{t+s} \right] \quad [039]$$

We now depart from Nickell's (1978) hypotheses, and, instead of strictly decreasing returns to scale, we assume constant returns. The production function $F(K_t, L_t)$ is then first-degree homogenous, which implies Euler's condition

$$F(K_t, L_t) = \frac{\partial F}{\partial K_t} K_t + \frac{\partial F}{\partial L_t} L_t \quad [040]$$

and, equivalently,

$$\frac{\partial F}{\partial K_t} K_t = F(K_t, L_t) - \frac{\partial F}{\partial L_t} L_t \quad [233]$$

$$p_t \frac{\partial F}{\partial K_t} K_t = p_t F(K_t, L_t) - p_t \frac{\partial F}{\partial L_t} L_t \quad [234]$$

Given first-order condition

$$p_t \frac{\partial F}{\partial L_t} = w_t \quad [016]$$

Euler's condition amounts to

$$p_t \frac{\partial F}{\partial K_t} K_t = p_t F(K_t, L_t) - w_t L_t \quad [041]$$

And equation [039] can be written

$$q_t K_{t+1} = \sum_{s=1}^{\infty} \frac{1}{(1+r)^s} [p_{t+s} F(K_{t+s}, L_{t+s}) - w_{t+s} L_{t+s} - q_{t+s} I_{t+s}] \quad [042]$$

that is,

$$\frac{\sum_{s=1}^{\infty} \frac{1}{(1+r)^s} [p_{t+s} F(K_{t+s}, L_{t+s}) - w_{t+s} L_{t+s} - q_{t+s} I_{t+s}]}{q_t K_{t+1}} = 1 \quad [043]$$

A2.3 BASIC DISCRETE-TIME MODEL : THE INTERTEMPORAL EQUILIBRIUM OF CAPITAL

Condition [017]

$$p_t \frac{\partial F}{\partial K_t} = (1+r) q_{t-1} - (1-\delta) q_t \quad [017]$$

amounts to [282]

$$q_t = \frac{1}{(1+r)} \left\{ (1-\delta) q_{t+1} + p_{t+1} \frac{\partial F}{\partial K_{t+1}} \right\} \quad [282]$$

By moving that equation forward to $t+1$, $t+2$, etc., there obtains

$$q_{t+1} = \frac{1}{(1+r)} \left\{ (1-\delta) q_{t+2} + p_{t+2} \frac{\partial F}{\partial K_{t+2}} \right\} \quad [286]$$

$$q_{t+2} = \frac{1}{(1+r)} \left\{ (1-\delta) q_{t+3} + p_{t+3} \frac{\partial F}{\partial K_{t+3}} \right\} \quad [287]$$

etc.

Then, by successive substitutions, we find

$$q_t = \frac{1}{(1+r)} \left\{ \begin{aligned} & \frac{(1-\delta)^3}{(1+r)^2} q_{t+3} + \frac{(1-\delta)^2}{(1+r)^2} p_{t+3} \frac{\partial F}{\partial K_{t+3}} \\ & + \frac{(1-\delta)}{(1+r)} p_{t+2} \frac{\partial F}{\partial K_{t+2}} \\ & + p_{t+1} \frac{\partial F}{\partial K_{t+1}} \end{aligned} \right\} \quad [044]$$

or, after rearranging

$$q_t = \frac{1}{(1+r)} \left\{ \begin{array}{l} \frac{(1-\delta)^s}{(1+r)^{s-1}} q_{t+s} + \frac{(1-\delta)^{s-1}}{(1+r)^{s-1}} p_{t+s} \frac{\partial F}{\partial K_{t+s}} \\ \vdots \\ \frac{(1-\delta)^2}{(1+r)^2} p_{t+3} \frac{\partial F}{\partial K_{t+3}} \\ + \frac{(1-\delta)}{(1+r)} p_{t+2} \frac{\partial F}{\partial K_{t+2}} \\ + p_{t+1} \frac{\partial F}{\partial K_{t+1}} \end{array} \right\} \quad [288]$$

$$q_t = \frac{1}{(1+r)} \left\{ \begin{array}{l} \frac{(1-\delta)^\theta}{(1+r)^{\theta-1}} q_{t+\theta} \\ + \sum_{s=1}^{\theta} \frac{(1-\delta)^{s-1}}{(1+r)^{s-1}} p_{t+s} \frac{\partial F}{\partial K_{t+s}} \end{array} \right\} \quad [289]$$

$$q_t = \frac{(1-\delta)^\theta}{(1+r)^\theta} q_\theta + \frac{1}{(1+r)} \sum_{s=1}^{\theta} \frac{(1-\delta)^{s-1}}{(1+r)^{s-1}} p_{t+s} \frac{\partial F}{\partial K_{t+s}} \quad [290]$$

$$q_t = \left(\frac{1-\delta}{1+r} \right)^\theta q_\theta + \frac{1}{(1+r)} \sum_{s=1}^{\theta} \left(\frac{1-\delta}{1+r} \right)^{s-1} p_{t+s} \frac{\partial F}{\partial K_{t+s}} \quad [045]$$

Make θ tend to infinity, and

$$q_t = \lim_{\theta \rightarrow \infty} \left(\frac{1-\delta}{1+r} \right)^\theta q_\theta + \frac{1}{(1+r)} \sum_{s=1}^{\infty} \left(\frac{1-\delta}{1+r} \right)^{s-1} p_{t+s} \frac{\partial F}{\partial K_{t+s}} \quad [046]$$

Impose the no-speculative-bubbles condition

$$\lim_{t \rightarrow \infty} \left(\frac{1-\delta}{1+r} \right)^t q_t = 0 \quad [050]$$

Now [046] becomes

$$q_t = \frac{1}{(1+r)} \sum_{s=1}^{\infty} \left(\frac{1-\delta}{1+r} \right)^{s-1} p_{t+s} \frac{\partial F}{\partial K_{t+s}} \quad [051]$$

A2.4 BASIC DISCRETE-TIME MODEL : THE USER COST OF CAPITAL WITH STATIONARY EXPECTATIONS

To simplify notation, let us denote the value of the marginal product of capital as

$$R_t = p_t \frac{\partial F}{\partial K_t} \quad [052]$$

Equation [051] becomes

$$q_t = \frac{1}{(1+r)} \sum_{s=0}^{\infty} \left(\frac{1-\delta}{1+r} \right)^s R_{t+s+1} \quad [053]$$

In that expression, replace R_{t+s+1} by its value such as expected at time t , and suppose that value to be the same for all s (stationary expectations) :

$$\tilde{R}_{t+s} = R_t, \forall s \geq 0 \quad [054]$$

where \tilde{R}_{t+s} is the value of R_{t+s} expected at time t .

The condition can be rewritten using the geometric series formula

$$\sum_{s=0}^{\infty} \left(\frac{1-\delta}{1+r} \right)^s = \left(\frac{1}{1 - \frac{1-\delta}{1+r}} \right) \quad [094]$$

$$\sum_{s=0}^{\infty} \left(\frac{1-\delta}{1+r} \right)^s = \frac{1}{\left(\frac{r+\delta}{1+r} \right)} \quad [095]$$

$$\sum_{s=0}^{\infty} \left(\frac{1-\delta}{1+r} \right)^s = \frac{1+r}{r+\delta} \quad [096]$$

Substitute [096] and [054] into [053], and

$$q_t = \frac{1}{(r+\delta)} R_t \quad [291]$$

$$R_t = (r+\delta) q_t \quad [055]$$

A2.5 MODEL WITH ADJUSTMENT COSTS : FIRST-ORDER OPTIMUM CONDITIONS

The firm maximizes the present value of its cash flow. If it is assumed that the discount rate is constant, the maximizing problem is therefore

$$\text{MAX } V = \sum_{t=0}^{\infty} \frac{1}{(1+r)^t} [p_t F(K_t, L_t) - w_t L_t - q_t I_t - C(I_t)] \quad [063]$$

$$\text{subject to : } I_t = K_{t+1} - K_t + \delta K_t = K_{t+1} - (1 - \delta)K_t \quad [014]$$

$$\text{and } K_0 = \bar{K}_0 \quad [005]$$

Substituting the adjustment cost function, the objective function becomes

$$\text{MAX } V = \sum_{t=0}^{\infty} \frac{1}{(1+r)^t} \left[p_t F(K_t, L_t) - w_t L_t - q_t I_t - q_t \frac{\gamma}{2} I_t^2 \right] \quad [292]$$

$$\text{MAX } V = \sum_{t=0}^{\infty} \frac{1}{(1+r)^t} \left[p_t F(K_t, L_t) - w_t L_t - q_t I_t \left(1 + \frac{\gamma}{2} I_t \right) \right] \quad [293]$$

Write the Lagrangian

$$\Lambda = \sum_{t=0}^{\infty} \frac{1}{(1+r)^t} \left\{ p_t F(K_t, L_t) - w_t L_t - q_t I_t \left(1 + \frac{\gamma}{2} I_t \right) + \lambda_t [I_t - K_{t+1} + (1 - \delta)K_t] \right\} - \mu (K_0 - \bar{K}_0) \quad [294]$$

Solving leads to the first-order conditions

$$\frac{\partial \Lambda}{\partial L_t} = \frac{1}{(1+r)^t} \left[p_t \frac{\partial F}{\partial L_t} - w_t \right] = 0 \quad [206]$$

$$\frac{\partial \Lambda}{\partial I_t} = \frac{1}{(1+r)^t} [q_t(-1 - \gamma I_t) + \lambda_t] = \frac{1}{(1+r)^t} [-q_t(1 + \gamma I_t) + \lambda_t] = 0 \quad [295]$$

$$\frac{\partial \Lambda}{\partial K_t} = \frac{1}{(1+r)^t} \left[p_t \frac{\partial F}{\partial K_t} - (1+r)\lambda_{t-1} + \lambda_t(1 - \delta) \right] = 0 \quad [276]$$

$$\frac{\partial \Lambda}{\partial \lambda_t} = \frac{1}{(1+r)^t} [I_t - K_{t+1} + (1 - \delta)K_t] \quad [209]$$

$$\frac{\partial \Lambda}{\partial \mu} = -(K_0 - \bar{K}_0) = 0 \quad [210]$$

Condition [295] amounts to

$$\lambda_{t+1} = q_t(1 + \gamma I_t) \quad [296]$$

Substituting λ_{t+1} and λ_t from the latter equation into [276], there results

$$p_t \frac{\partial F}{\partial K_t} - (1+r)q_{t-1}(1 + \gamma I_{t-1}) + q_t(1 + \gamma I_t)(1 - \delta) = 0 \quad [064]$$

The first-order conditions become

$$p_t \frac{\partial F}{\partial L_t} = w \quad [016]$$

$$p_t \frac{\partial F}{\partial K_t} = (1+r)q_{t-1}(1+\gamma I_{t-1}) - q_t(1+\gamma I_t)(1-\delta) \quad [064]$$

$$I_t = K_{t+1} - (1-\delta)K_t \quad [014]$$

$$K_0 = \bar{K}_0 \quad [005]$$

A2.6 MODEL WITH ADJUSTMENT COSTS : TOBIN'S q

Denote the value of the marginal productivity of capital as

$$R_t = p_t \frac{\partial F}{\partial K_t} \quad [052]$$

Develop [066]²⁷ :

$$p_t \frac{\partial F}{\partial K_t} = (1+r)Q_{t-1} - (1-\delta)Q_t \quad [066]$$

$$R_t = (1+r)Q_{t-1} - (1-\delta)Q_t \quad [297]$$

$$(1+r)Q_{t-1} = (1-\delta)Q_t + R_t \quad [298]$$

$$(1+r)Q_t = (1-\delta)Q_{t+1} + R_{t+1} \quad [299]$$

$$Q_t = \frac{1}{(1+r)} [(1-\delta)Q_{t+1} + R_{t+1}] \quad [300]$$

$$Q_t K_{t+1} = \frac{1}{(1+r)} [(1-\delta)Q_{t+1} K_{t+1} + R_{t+1} K_{t+1}] \quad [073]$$

Now, the accumulation constraint

$$I_t = K_{t+1} - (1-\delta)K_t \quad [014]$$

amounts to

$$(1-\delta)K_t = K_{t+1} - I_t \quad [030]$$

Substituting into [073] yields

$$Q_t K_{t+1} = \frac{1}{(1+r)} (Q_{t+1} (K_{t+2} - I_{t+1}) + R_{t+1} K_{t+1}) \quad [301]$$

²⁷ The development that follows is parallel to Hayashi's (1982), as reproduced in Nabil Annabi, *Les MEGC avec anticipations rationnelles : introduction*, slide presentation, March 2003; see slides No. 38 and following.

$$Q_t K_{t+1} = \frac{1}{(1+r)} (Q_{t+1} K_{t+2} - Q_{t+1} I_{t+1} + R_{t+1} K_{t+1}) \quad [302]$$

$$Q_t K_{t+1} = \frac{1}{(1+r)} (R_{t+1} K_{t+1} - Q_{t+1} I_{t+1} + Q_{t+1} K_{t+2}) \quad [074]$$

where the term $Q_{t+1} K_{t+2}$ can be replaced by its expression according to that very same relation.

Then

$$Q_t K_{t+1} = \frac{1}{(1+r)} \left[\frac{R_{t+1} K_{t+1} - Q_{t+1} I_{t+1}}{(1+r)} + \frac{1}{(1+r)} (R_{t+2} K_{t+2} - Q_{t+2} I_{t+2} + Q_{t+2} K_{t+3}) \right] \quad [075]$$

By successive substitutions, we arrive at

$$Q_t K_{t+1} = \sum_{s=1}^{\infty} \frac{1}{(1+r)^s} [R_{t+s} K_{t+s} - Q_{t+s} I_{t+s}] + \lim_{s \rightarrow \infty} \left(\frac{1}{(1+r)^s} Q_{t+s} K_{t+s+1} \right) \quad [076]$$

where the last term is null, by virtue of the transversality condition

$$\lim_{s \rightarrow \infty} \left(\frac{1}{(1+r)^s} Q_{t+s} K_{t+s+1} \right) = 0 \quad [077]$$

Therefore,

$$Q_t K_{t+1} = \sum_{s=1}^{\infty} \frac{1}{(1+r)^s} [R_{t+s} K_{t+s} - Q_{t+s} I_{t+s}] \quad [078]$$

We assume constant returns to scale. The production function $F(K_t, L_t)$ is then first-degree homogenous, which implies Euler's condition

$$F(K_t, L_t) = \frac{\partial F}{\partial K_t} K_t + \frac{\partial F}{\partial L_t} L_t \quad [040]$$

and, equivalently,

$$\frac{\partial F}{\partial K_t} K_t = F(K_t, L_t) - \frac{\partial F}{\partial L_t} L_t \quad [233]$$

$$\rho_t \frac{\partial F}{\partial K_t} K_t = \rho_t F(K_t, L_t) - \rho_t \frac{\partial F}{\partial L_t} L_t \quad [234]$$

Given the first-order condition

$$\rho_t \frac{\partial F}{\partial L_t} = w_t \quad [016]$$

Euler's condition amounts to

$$p_t \frac{\partial F}{\partial K_t} K_t = p_t F(K_t, L_t) - w_t L_t \quad [041]$$

or

$$R_t K_t = p_t F(K_t, L_t) - w_t L_t \quad [079]$$

Now, replace $R_{t+s} K_{t+s}$ in [078] :

$$Q_t K_{t+1} = \sum_{s=1}^{\infty} \frac{1}{(1+r)^s} [p_{t+s} F(K_{t+s}, L_{t+s}) - w_{t+s} L_{t+s} - Q_{t+s} I_{t+s}] \quad [080]$$

Again, recall that

$$Q_t = \frac{\partial}{\partial I_t} \left[q_t I_t \left(1 + \frac{\gamma}{2} I_t \right) \right] = q_t (1 + \gamma I_t) \quad [065]$$

and make [078] more explicit :

$$Q_t K_{t+1} = \sum_{s=1}^{\infty} \frac{1}{(1+r)^s} [p_{t+s} F(K_{t+s}, L_{t+s}) - w_{t+s} L_{t+s} - q_{t+s} I_{t+s} (1 + \gamma I_{t+s})] \quad [303]$$

$$Q_t K_{t+1} = \sum_{s=1}^{\infty} \frac{1}{(1+r)^s} \left[\begin{array}{l} p_{t+s} F(K_{t+s}, L_{t+s}) - w_{t+s} L_{t+s} \\ - q_{t+s} I_{t+s} \left(1 + \frac{\gamma}{2} I_{t+s} \right) - q_{t+s} I_{t+s} \left(\frac{\gamma}{2} I_{t+s} \right) \end{array} \right] \quad [304]$$

that is,

$$\frac{\sum_{s=1}^{\infty} \frac{1}{(1+r)^s} \left[\begin{array}{l} p_{t+s} F(K_{t+s}, L_{t+s}) - w_{t+s} L_{t+s} \\ - q_{t+s} I_{t+s} \left(1 + \frac{\gamma}{2} I_{t+s} \right) - q_{t+s} I_{t+s} \left(\frac{\gamma}{2} I_{t+s} \right) \end{array} \right]}{Q_t K_{t+1}} = 1 \quad [305]$$

$$\frac{\sum_{s=1}^{\infty} \frac{1}{(1+r)^s} \left[\begin{array}{l} p_{t+s} F(K_{t+s}, L_{t+s}) - w_{t+s} L_{t+s} \\ - q_{t+s} I_{t+s} \left(1 + \frac{\gamma}{2} I_{t+s} \right) \end{array} \right]}{Q_t K_{t+1}} = 1 + \frac{\sum_{s=1}^{\infty} \frac{1}{(1+r)^s} \left[q_{t+s} I_{t+s} \left(\frac{\gamma}{2} I_{t+s} \right) \right]}{Q_t K_{t+1}} \quad [081]$$

A2.7 MODEL WITH ADJUSTMENT COSTS : THE INTERTEMPORAL EQUILIBRIUM OF CAPITAL

The condition

$$p_t \frac{\partial F}{\partial K_t} = (1+r)Q_{t-1} - (1-\delta)Q_t \quad [066]$$

amounts to

$$(1 - \delta)Q_t = (1 + r)Q_{t-1} - p_t \frac{\partial F}{\partial K_t} \quad [306]$$

$$Q_t = \frac{(1 + r)}{(1 - \delta)} Q_{t-1} - \frac{1}{(1 - \delta)} p_t \frac{\partial F}{\partial K_t} \quad [084]$$

Optimum is achieved when investment in period t is at the point where its marginal cost Q_t (which grows with I_t) satisfies condition [084].

Moving that condition forward one period results in

$$Q_{t+1} = \frac{(1 + r)}{(1 - \delta)} Q_t - \frac{1}{(1 - \delta)} p_{t+1} \frac{\partial F}{\partial K_{t+1}} \quad [307]$$

which can be written as

$$Q_t = \frac{1}{(1 + r)} \left\{ (1 - \delta) Q_{t+1} + p_{t+1} \frac{\partial F}{\partial K_{t+1}} \right\} \quad [085]$$

Moving that equation forward to $t+1$, $t+2$, etc., we get

$$Q_{t+1} = \frac{1}{(1 + r)} \left\{ (1 - \delta) Q_{t+2} + p_{t+2} \frac{\partial F}{\partial K_{t+2}} \right\} \quad [308]$$

$$Q_{t+2} = \frac{1}{(1 + r)} \left\{ (1 - \delta) Q_{t+3} + p_{t+3} \frac{\partial F}{\partial K_{t+3}} \right\} \quad [309]$$

etc.

Then, by successive substitutions, we find

$$Q_t = \frac{1}{(1 + r)} \left\{ \begin{aligned} & \frac{(1 - \delta)^3}{(1 + r)^2} Q_{t+3} + \frac{(1 - \delta)^2}{(1 + r)^2} p_{t+3} \frac{\partial F}{\partial K_{t+3}} \\ & + \frac{(1 - \delta)}{(1 + r)} p_{t+2} \frac{\partial F}{\partial K_{t+2}} \\ & + p_{t+1} \frac{\partial F}{\partial K_{t+1}} \end{aligned} \right\} \quad [086]$$

that is, after rearranging,

$$Q_t = \frac{1}{(1+r)} \left\{ \begin{array}{l} \frac{(1-\delta)^s}{(1+r)^{s-1}} Q_{t+s} + \frac{(1-\delta)^{s-1}}{(1+r)^{s-1}} p_{t+s} \frac{\partial F}{\partial K_{t+s}} \\ \vdots \\ + \frac{(1-\delta)^2}{(1+r)^2} p_{t+3} \frac{\partial F}{\partial K_{t+3}} \\ + \frac{(1-\delta)}{(1+r)} p_{t+2} \frac{\partial F}{\partial K_{t+2}} \\ + p_{t+1} \frac{\partial F}{\partial K_{t+1}} \end{array} \right\} \quad [310]$$

With the no-speculative-bubbles condition,

$$\lim_{t \rightarrow \infty} \left(\frac{1-\delta}{1+r} \right)^t Q_t = 0 \quad [087]$$

we finally get

$$Q_t = \frac{1}{(1+r)} \sum_{s=1}^{\infty} \left(\frac{1-\delta}{1+r} \right)^{s-1} p_{t+s} \frac{\partial F}{\partial K_{t+s}} \quad [088]$$

Given

$$R_t = p_t \frac{\partial F}{\partial K_t} \quad [052]$$

we can write

$$Q_t = \frac{1}{(1+r)} \sum_{s=1}^{\infty} \left(\frac{1-\delta}{1+r} \right)^{s-1} R_{t+s} \quad [089]$$

or, equivalently,

$$Q_t = \frac{1}{(1+r)} \sum_{s=0}^{\infty} \left(\frac{1-\delta}{1+r} \right)^s R_{t+s+1} \quad [090]$$

**PART TWO :
SAVINGS**

Introduction

In the static version of the Ministère des Finances du Québec CGE model (MÉGFQ), household savings are a fixed fraction of disposable income. Such a specification is usual in CGE models, including in dynamic financial models. Of all the models reviewed by Thissen (1999)²⁸, only those of Fargeix and Sadoulet (1994) and of Lewis (1994) have savings depend on the interest rate. In all the other models, savings are a fixed fraction of income : the rates of interest influence only portfolio allocation. The same is true for the models of Agénor (2003), of Bchir *et al.* (2002) and of Decreux (1999).

Collange's (1993) model is distinctive in that savings are a fixed proportion, not only of income, but rather of the difference between disposable income and the gain in wealth due to the revaluation of previous wealth. If inflation is positive, wealth is devalued, the gain in wealth is negative, and households increase their savings in order to partly compensate for the fall in their real wealth.

Jung and Thorbecke (2001) write : « Savings are determined by income, and investments by the interest rate (in the savings equation in the model we assume that savings are insensitive to changes in the interest rate, consistent with the observed trends in the two economies) ». Yet, in their equation (38), in appendix II, savings are linked to the interest rate by a constant elasticity function :

$$SH_h = s_h (1+r)^{\beta_{sh}} YD_h \quad [311]$$

where

SH_h are the savings of household h ;

s_h is the marginal propensity to save of household h ;

r is the rate of interest;

YD_h is the disposable income of household h ;

β_h is the elasticity parameter of household h savings.

²⁸ Thissen reviews the following models : Feltenstein (1980, 1984, 1995), Bourguignon, Branson and de Melo (1989, 1992), Rosensweig and Taylor (1990), Easterly (1990), Fargeix and Sadoulet (1990, 1994) Lewis (1992, 1994), Decaluwé and Nsengiyumva (1994), Souissi and Decaluwé (1997), Yeldan (1997), Vos (1998), and Naastepad (1998). Let us also mention Pereira and Shoven (1988), who surveyed 11 dynamic models; but these are intertemporal dynamic models (at least with respect to consumer behavior), rather than recursive dynamic models of the kind discussed here.

So it appears that Jung and Thorbecke have set parameter β_h at zero, in which case savings are simply proportional to disposable income. Anyway, Jung and Thorbecke's equation (39) does not seem to be based on explicit theoretical foundations.

in Mensbrugghe (1994, 2003) and in Beghin *et al.* (1996), household savings are determined in an *Extended Linear Expenditure System* (ELES). We shall return to that specification later.

But why not stick to the usual specification, where household savings are a fixed proportion of disposable income? We have several reasons. The first is that in observed SAM's (and in particular, in the SAM underlying the Ministère des Finances du Québec CGE model), some household types have negative savings. It follows that the average propensity to save is also negative, so that, with the usual specification, an increase in income *reduces* the amount of savings (more precisely, increases the amount of negative savings). Obviously, such anomaly could be eliminated by replacing the usual savings function by a linear savings function with a calibrated intercept and a positive marginal (rather than average) propensity to be estimated econometrically :

$$SM_{men} = SMO_{men} + \psi_{men} YDM_{men} \quad [312]$$

where

SM_{men} are the savings of household type men ;

SMO_{men} is the intercept of the savings function of household type men ;

ψ_{men} is the marginal propensity to save of household type men ;

YDM_{men} is the disposable income of household type men .

A second reason to revise the savings model is that, in its usual form, savings are completely insensitive to the rate of interest.

A third reason, finally, is the way in which savings influence the opportunity cost of leisure in the endogenous labor supply model, as it is specified in Decaluwé *et al.* (2005, p. 17). There, the opportunity cost of leisure is

$$PCTL_{l,men,rg} = (1 - \psi_{men}) \left[1 - \left(\sum_{gvt} tytemi_{gvt,men}^{TD} + \sum_{gvt} \sum_{prr} tytemi_{gvt,prr}^{TR} \right) \right] (1 - TCHO_{l,rg}) w_{l,rg}$$

[313]

where

$PCTL_{men,l,rg}$ is the opportunity cost of leisure corresponding to the labor supply of occupational category l by type men households in region rg ;
 ψ_{men} is the marginal propensity to save of type men households
 $tytemi_{gvt,men}^{td}$ is the marginal income tax rate of men type households by government gvt ;
 $tytemi_{gvt,pr}^{TR}$ is the implicit marginal labor income tax rate due to the reduction of type pr transfers by government gvt ;
 pr is an index referring to transfers that are reducible depending on income;
 $TCHO_{l,rg}$ is the rate of unemployment of occupational category l in region rg ;
 $w_{l,rg}$ is the wage rate of occupational category l in region rg .

The opportunity cost of leisure for member l,rg in household men is equal to the mathematical expectation of the wage rate of occupational category l , net of income tax and savings. In that model, indeed, savings, no more than taxes, contribute to the household's utility. It must be recognized that it is a restrictive hypothesis, especially in the context of a dynamic model.

The rest of this part of the document consists in a presentation of the « Super-Extended Linear Expenditure System » (SELES) model. The presentation is in stages. First, we recall the specification of the classical ELES model. Then, we propose a formulation for the price of future consumption which takes into account returns on savings. But, even with that new definition of the price of future consumption, the amount of savings remains insensitive to the rate of return when the utility function is Stone-Geary as in the ELES. For that reason, we introduce a minimal quantity of future consumption in the utility function; savings then become sensitive to the rate of return. The final step in the development is to make labor supply endogenous by introducing leisure in the utility function. In the end, the model's parameter calibration strategy is sketched.

1. The Extended Linear Expenditure System (ELES)

Let us begin with a simplified presentation of the ELES model such as it is applied by Mensbrugghe (1994, 2003) and by Beghin *et al.* (1996).

The consumer's problem is to maximize a Stone-Geary utility function, extended to include savings, that is, future consumption :

$$\ln U = \sum_i \gamma_i \ln(C_i - C_i^{MIN}) + \gamma^F \ln\left(\frac{S}{PAF}\right) \quad [314]$$

$$\text{subject to } \sum_i P_i C_i + S = YD \quad [315]$$

where

C_i is the quantity consumed of commodity i ;

C_i^{MIN} is the minimum consumption of commodity i ;

S is savings;

PAF is the expected future price of consumption, that is, the appropriate price index to convert nominal savings S into real future consumption;

P_i is the price of commodity i ;

YD is disposable income.

In Mensbrugge, as in Beghin *et al.*, PAF is simply the consumer price index.

Of course, the utility function parameters respect

$$\sum_i \gamma_i + \gamma^F = 1 \quad [316]$$

The optimization problem Lagrangian is

$$\Lambda = \sum_i \gamma_i \ln(C_i - C_i^{MIN}) + \gamma^F \ln\left(\frac{S}{PAF}\right) - \lambda \left(\sum_i P_i C_i + S - YD \right) \quad [317]$$

Whence, first-order conditions

$$\frac{\partial \Lambda}{\partial C_i} = \frac{\gamma_i}{(C_i - C_i^{MIN})} - \lambda P_i = 0, \text{ that is, } (C_i - C_i^{MIN}) = \frac{\gamma_i}{\lambda P_i} \quad [318]$$

$$\frac{\partial \Lambda}{\partial S} = \frac{\gamma^F}{\left(\frac{S}{PAF}\right)} \frac{1}{PAF} - \lambda = \frac{\gamma^F}{S} - \lambda = 0, \text{ that is, } \frac{\gamma^F}{S} = \lambda \quad [319]$$

Next, substituting λ from [319] into [318],

$$(C_i - C_i^{MIN}) = \frac{\gamma_i}{\lambda P_i} = \gamma_i \frac{S}{\gamma^F P_i} = S \frac{\gamma_i}{\gamma^F P_i}, \text{ ou } C_i = C_i^{MIN} + S \frac{\gamma_i}{\gamma^F P_i} \quad [320]$$

The sum over all commodities of demand [320] is

$$\sum_i P_i C_i = \sum_i P_i C_i^{MIN} + \sum_i S \frac{\gamma_i}{\gamma^F} = \sum_i P_i C_i^{MIN} + \frac{1 - \gamma^F}{\gamma^F} S \quad [321]$$

Substitute [321] in budget constraint [315] :

$$YD = \sum_i P_i C_i + S = \sum_i P_i C_i^{MIN} + \frac{1-\gamma^F}{\gamma^F} S + S = \sum_i P_i C_i^{MIN} + \frac{1}{\gamma^F} S \quad [322]$$

Given [322] and the definition of supernumerary income $CSUP$,

$$CSUP = YD - \sum_i P_i C_i^{MIN} = \frac{1}{\gamma^F} S \quad [323]$$

Substituting [323] into [320], we obtain the demand equations

$$C_i = C_i^{MIN} + \gamma_i \frac{CSUP}{P_i} \quad [324]$$

Then, inverting [323], we get the savings function

$$S = \gamma^F CSUP \quad [325]$$

Savings are a constant fraction of supernumerary income, and, as can be seen in [323], the latter is independent of the rate of return on savings r . It follows that savings are insensitive to the rate of return.

2. The price of future consumption and the rate of return on savings

In the classical ELES model just presented, savings intervene in the utility function as though the consumer purchased a quantity $\frac{S}{PAF}$ of goods to then store them (without cost) for the future. That ignores the fact that savings generate returns...

With stationary expectations, an amount S of savings will yield a recurrent income equal to $r \times S$, where r is the real, net of taxes, rate of return on savings. In every future period, that recurring income of $r \times S$ will make it possible to acquire a quantity $\frac{rS}{PAF}$ of a composite good;

PAF is the expected future price of the composite consumption good (with stationary expectations, that price is an appropriate index computed from current prices, as in Mensbrugghe, 1994, 2003, and in Beghin *et al.*, 1996). With a psychological discount rate (or rate of time-preference) equal to f , the quantity CF of future consumption made possible by an amount S of savings is equal to the present value of the future consumption which that recurring income will sustain :

$$CF = \sum_{t=1}^{\infty} \frac{r S}{(1+f)^t PAF} = \frac{r}{f} \frac{S}{PAF} \quad [326]$$

that is,

$$S = \left(\frac{f}{r} PAF \right) CF \quad [327]$$

Amount S is the expenditure thanks to which the household can acquire future consumption of CF at price $\frac{f}{r} PAF$. If it is assumed that the rate of time-preference f is constant, then the price of future consumption is inversely proportional to the rate of return on savings : the higher the rate of return, the less it costs to acquire future consumption.

Just like in the classical ELES, however, the price of future consumption is absent from savings function [325], so that, despite the fact that the price is inversely proportional to r , savings remain insensitive to the rate of return.

3. Savings made sensitive to the rate of return

The lack of sensitivity of savings to the rate of return and to the price of future consumption comes from the fact that the Stone-Geary utility function is nothing but a Cobb-Douglas function of supernumerary income, a functional form with well-known characteristics, one of which is the independence of budget shares with respect to prices. It follows that, if savings (future consumption) are included in the utility function without a minimum level of future consumption, then its share in supernumerary income is constant, and the amount of savings is independent from the rate of return.

But savings can be made sensitive to the rate of return and to the expected future price of goods by introducing a minimum level of future consumption in the ELES. The utility function then becomes

$$\ln U = \sum_i \gamma_i \ln(C_i - C_i^{MIN}) + \gamma^F \ln(CF - CF^{MIN}) \quad [328]$$

with budget constraint

$$\sum_i P_i C_i + \frac{f PAF}{r} CF = YD \quad [329]$$

where the second term is savings as defined in [327].

The Lagrangian of the optimization problem is

$$\Lambda = \sum_i \gamma_i \ln(C_i - C_i^{MIN}) + \gamma^F \ln(CF - CF^{MIN}) - \lambda \left(\sum_i P_i C_i + \frac{f PAF}{r} CF - YD \right) \quad [330]$$

Whence, first-order conditions

$$\frac{\partial \Lambda}{\partial C_i} = \left(\frac{\gamma_i}{C_i - C_i^{MIN}} \right) - \lambda P_i = 0, \text{ c'est-à-dire } (C_i - C_i^{MIN}) = \frac{\gamma_i}{\lambda P_i} \quad [331]$$

$$\frac{\partial \Lambda}{\partial CF} = \left(\frac{\gamma^F}{CF - CF^{MIN}} \right) - \lambda \frac{f PAF}{r} = 0, \text{ c'est-à-dire } \lambda = \frac{\gamma^F}{\left(\frac{f PAF}{r} \right) (CF - CF^{MIN})} \quad [332]$$

Next, substituting λ from [332] into [331],

$$(C_i - C_i^{MIN}) = \frac{\gamma_i}{\lambda P_i} = \gamma_i \frac{\left(\frac{f PAF}{r} \right) (CF - CF^{MIN})}{\gamma^F P_i},$$

$$\text{or } C_i = C_i^{MIN} + \gamma_i \frac{\left(\frac{f PAF}{r} \right) (CF - CF^{MIN})}{\gamma^F P_i} \quad [333]$$

The sum of demand equation [333] over all goods is

$$\begin{aligned} \sum_i P_i C_i &= \sum_i P_i C_i^{MIN} + \left(\sum_i \gamma_i \right) \frac{\left(\frac{f PAF}{r} \right) (CF - CF^{MIN})}{\gamma^F} \\ &= \sum_i P_i C_i^{MIN} + \frac{1 - \gamma^F}{\gamma^F} \left(\frac{f PAF}{r} \right) (CF - CF^{MIN}) \end{aligned} \quad [334]$$

Substitute [334] into budget constraint [329] :

$$YD = \sum_i P_i C_i^{MIN} + \left(\frac{f PAF}{r} \right) CF^{MIN} + \frac{1}{\gamma^F} \left(\frac{f PAF}{r} \right) (CF - CF^{MIN}) \quad [335]$$

Demonstration :

$$YD = \sum_i P_i C_i + \frac{f PAF}{r} CF \quad [329]$$

$$YD = \left[\sum_i P_i C_i^{MIN} + \frac{1 - \gamma^F}{\gamma^F} \left(\frac{f PAF}{r} \right) (CF - CF^{MIN}) \right] + \frac{f PAF}{r} CF \quad [336]$$

$$YD = \left[\sum_i P_i C_i^{MIN} + \frac{1-\gamma^F}{\gamma^F} \left(\frac{f PAF}{r} \right) (CF - CF^{MIN}) \right] + \frac{f PAF}{r} CF - \frac{f PAF}{r} CF^{MIN} + \frac{f PAF}{r} CF^{MIN} \quad [337]$$

$$YD = \left[\sum_i P_i C_i^{MIN} + \frac{1-\gamma^F}{\gamma^F} \left(\frac{f PAF}{r} \right) (CF - CF^{MIN}) \right] + \frac{f PAF}{r} (CF - CF^{MIN}) + \frac{f PAF}{r} CF^{MIN} \quad [338]$$

$$YD = \sum_i P_i C_i^{MIN} + \left(\frac{f PAF}{r} \right) CF^{MIN} + \left[\frac{1-\gamma^F}{\gamma^F} \left(\frac{f PAF}{r} \right) + \left(\frac{f PAF}{r} \right) \right] (CF - CF^{MIN}) \quad [339]$$

$$YD = \sum_i P_i C_i^{MIN} + \left(\frac{f PAF}{r} \right) CF^{MIN} + \frac{1}{\gamma^F} \left(\frac{f PAF}{r} \right) (CF - CF^{MIN}) \quad [335]$$

Given [335] and the [re-]definition of supernumerary income $CSUP$ with a minimum of future consumption,

$$CSUP = YD - \sum_i P_i C_i^{MIN} - \left(\frac{f PAF}{r} \right) CF^{MIN} = \frac{1}{\gamma^F} \left(\frac{f PAF}{r} \right) (CF - CF^{MIN}) \quad [340]$$

Substituting [340] into [333], we obtain demand equations

$$C_i = C_i^{MIN} + \gamma_i \frac{CSUP}{P_i} \quad [324]$$

Then, inverting [340], we get the savings function

$$S = \left(\frac{f PAF}{r} \right) CF = \left(\frac{f PAF}{r} \right) CF^{MIN} + \gamma^F CSUP \quad [341]$$

Commodity demand equations [324] are formally identical to those of the classical ELES model, but here, the definition of $CSUP$ includes a minimum quantity of future consumption CF^{MIN} , as evidenced by [340].

The derivative of S with respect to r in [341] is

$$\frac{\partial S}{\partial r} = \frac{\partial S}{\partial \left(\frac{f PAF}{r} \right)} \frac{\partial \left(\frac{f PAF}{r} \right)}{\partial r} = -\frac{1}{r^2} \frac{\partial S}{\partial \left(\frac{f PAF}{r} \right)} \quad [342]$$

$$\frac{\partial S}{\partial r} = -\frac{1}{r^2} \left[CF^{MIN} + \gamma^F \frac{\partial CSUP}{\partial \left(\frac{f PAF}{r} \right)} \right] = -\frac{1}{r^2} (1 - \gamma^F) CF^{MIN} \quad [343]$$

As for the derivative of S with respect to PAF , it is

$$\frac{\partial S}{\partial PAF} = \frac{\partial S}{\partial \left(\frac{f PAF}{r} \right)} \frac{\partial \left(\frac{f PAF}{r} \right)}{\partial PAF} = \frac{f}{r} \frac{\partial S}{\partial \left(\frac{f PAF}{r} \right)} = \frac{f}{r} (1 - \gamma^F) CF^{MIN} \quad [344]$$

The effect on savings of an increase in the rate of return is positive or negative, depending on whether CF^{MIN} is negative or positive; contrariwise, the effect of an increase of expected future commodity prices is of the same sign as CF^{MIN} . In other words, since the price of future consumption varies in inverse proportion to rate of return r , while it is proportional to the expected price of goods, the derivative of the demand for future consumption relative to its price is of the same sign as CF^{MIN} .

Note that, since r is the rate of return net of income taxes, an increase in taxes diminishes the rate of return, *ceteris paribus*, and thus raises the price of future consumption, which makes savings increase for households for which minimum future consumption is positive, and decrease for households for which minimum future consumption is negative.

Overall, if a household borrows when its income is low (negative CF^{MIN} , a possibility that does not seem unlikely for less-well-off households), then its savings increase (its debt diminishes) when the rate of return rises and the price of future consumption falls. That happens because the first term of demand function [341], which is negative, diminishes in absolute value, and because that fall is greater in absolute value than the decrease in the second term, which is positive : the drain on savings from the « basic amount of borrowing », that is, the absolute value of $\left(\frac{f PAF}{r} \right) CF^{MIN}$, diminishes, which simultaneously reduces supernumerary income

$CSUP$ by the same amount, but the first effect dominates the second, because the coefficient of $CSUP$ in [341], γ^F , is greater than 1.

Inversely, if a household is sufficiently rich not to have to borrow, even when its income is low, then the cost of the minimum future consumption $\left(\frac{f PAF}{r}\right) CF^{MIN}$ falls as r increases, and that is not compensated by the rise in discretionary savings $\gamma^F CSUP$.

One might question the meaning of a negative level of minimum future consumption. But recall that the minimum consumption of a commodity in the Stone-Geary utility function is not to be taken literally to be a « vital minimum »; rather, it is a quantity below which consumption of the good may not generate utility. In the case of future consumption, it is not implausible that a household which must borrow to live begin to feel a certain satisfaction (positive utility) when its borrowings are below a certain amount : in other words, even if savings are negative and future consumption is sacrificed to the present, a program where the sacrifice of future consumption is below some threshold nevertheless yields utility.

4. The SELES model : savings sensitive to the rate of return, with endogenous labor supply

In the modified ELES model described so far, the supply of labor is not endogenous. So let us complete our model by introducing leisure in the utility function.

Let the utility function to be maximized be

$$\ln U = \sum_i \gamma_i \ln(C_i - C_i^{MIN}) + \gamma^L \ln(L - L^{MIN}) + \gamma^F \ln(CF - CF^{MIN}) \quad [345]$$

with budget constraint

$$\sum_i P_i C_i + w L + PF CF = y + w LS^{MAX} \quad [346]$$

where

L is the consumption of leisure;

L^{MIN} is the minimum consumption of leisure;

w is the price of leisure (wage rate, net of income tax);

$PF = \frac{f}{r} PAF$ is the price of future consumption;

y is non-labor disposable income;

LS^{MAX} is the maximum labor time.

The utility function parameters respect

$$\sum_i \gamma_i + \gamma^L + \gamma^F = 1 \quad [347]$$

Naturally, savings are

$$S = PF CF = \left(\frac{f}{r} PAF \right) CF \quad [348]$$

The Lagrangian of the optimization problem is

$$\begin{aligned} \Lambda = & \sum_i \gamma_i \ln(C_i - C_i^{MIN}) + \gamma^L \ln(L - L^{MIN}) + \gamma^F \ln(CF - CF^{MIN}) \\ & - \lambda \left(\sum_i P_i C_i + wL + PF CF - y - w LS^{MAX} \right) \end{aligned} \quad [349]$$

Define the **integral** supernumerary income as it should be when $CF^{MIN} \neq 0$:

$$CSUPINT = y + w LS^{MAX} - \sum_i P_i C_i^{MIN} - wL^{MIN} - PF CF^{MIN} \quad [350]$$

The demand equations are

$$P_i (C_i - C_i^{MIN}) = \gamma_i CSUPINT \quad [351]$$

$$PF (CF - CF^{MIN}) = \gamma^F CSUPINT \quad [352]$$

$$w (L - L^{MIN}) = \gamma^L CSUPINT \quad [353]$$

(see details in Appendix B).

The supply of labor is then

$$LS = LS^{MAX} - L = LS^{MAX} - L^{MIN} - \gamma^L \frac{CSUPINT}{w} \quad [354]$$

Define the parameter

$$MAXHEURES = LS^{MAX} - L^{MIN} \quad [355]$$

and rewrite

$$LS = LS^{MAX} - L = MAXHEURES - \gamma^L \frac{CSUPINT}{w} \quad [356]$$

Also note that the integral supernumerary income [350], written in terms of *MAXHEURES*, is

$$CSUPINT = y + w MAXHEURES - \sum_i P_i C_i^{MIN} - PF CF^{MIN} \quad [357]$$

So it is absolutely possible that savings be sensitive to the rate of return with endogenous labor supply.

5. Summary and calibration considerations

In many CGE models, savings are simply proportional to disposable income. In the Extended Linear Expenditure System (ELES), applied by Mensbrugghe (1994, 2003) and by Beghin *et al.* (1996), savings contribute to household utility, but remains nevertheless independent of the rate of return.

The relationship between savings and real future consumption depends on the expected prices of consumption goods. But the authors mentioned above can be criticized for not taking into account the fact that the price of future consumption also depends on the rate of return on savings : the higher the rate of return, the lower the amount to be saved for a given quantity of future consumption. However, even if the price of future consumption is formulated in such a way as to take account of the return on savings, the latter remain insensitive to the rate of interest.

But the ELES model can be modified in order to make savings sensitive to the rate of interest. It is only necessary to assume there is a minimum quantity of future consumption, just as, in the linear expenditure system, there is a minimum quantity of every good. Such a specification also allows to represent the reality of certain categories of households which have negative savings. It is also consistent with the idea that, below some income threshold, households will tend to borrow or liquidate part of their wealth.

Questions arise concerning the possibility of calibrating the parameters of the SELES model. The calibration strategy consists of four steps :

1. determine the value of the marginal propensity to consume leisure γ^L ;
2. econometrically estimate the marginal propensity to save γ^F ;
3. calibrate the parameters of the current consumption goods demand functions;
4. calibrate parameters LS^{MAX} and CF^{MIN} .

Knowing the values of total disposable income and of labor income in the SAM, the value of parameter γ^L can be determined from the income elasticity of labor supply, the value of which is exogenously chosen, usually from a survey of the literature.

As for marginal propensity to save γ^F , equation [341] can be reduced to a relation of the form

$$\text{Savings} = \text{Constant} + b \times \text{Disposable income}$$

If it is assumed that the parameters of that relation are identical for all households in a given category, then they can be estimated econometrically by means of microdata. In principle, one could also obtain the value of the constant in the savings function of each household category by multiplying the value of the intercept estimated from the microdata by the number of households in the category. In practice however, it is highly improbable that the savings function obtained in this way be consistent with the SAM. So PF and CF^{MIN} will have to be calibrated after γ^F has been estimated, by inverting the savings function [341].

Once the values of γ^L and γ^F have been determined, it is possible to rearrange the commodity demand equations so that they reassume their classical LES form. Then the calibration procedure described in Annabi *et al.* (2006) can be applied.

Finally, parameter LS^{MAX} is calibrated from labor supply function [356], all other elements of which are known values at this point, given [357]²⁹.

²⁹ See Annabi (2003).

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Appendix B : Derivation of the demand functions of the SELES model

Let the utility function to be maximized be

$$\ln U = \sum_i \gamma_i \ln(C_i - C_i^{MIN}) + \gamma^L \ln(L - L^{MIN}) + \gamma^F \ln(CF - CF^{MIN}) \quad [345]$$

under budget constraint

$$\sum_i P_i C_i + wL + PF CF = y + w LS^{MAX} \quad [346]$$

where

L is the consumption of leisure;

L^{MIN} is the minimum consumption of leisure;

w is the price of leisure (wage rate, net of income tax);

$PF = \frac{f}{r} PAF$ is the price of future consumption;

y is non-labor disposable income;

LS^{MAX} is the maximum labor time.

The utility function parameters respect

$$\sum_i \gamma_i + \gamma^L + \gamma^F = 1 \quad [347]$$

Naturally, savings are

$$S = PF CF = \left(\frac{f}{r} PAF \right) CF \quad [348]$$

The Lagrangian of the optimization problem is

$$\begin{aligned} \Lambda = & \sum_i \gamma_i \ln(C_i - C_i^{MIN}) + \gamma^L \ln(L - L^{MIN}) + \gamma^F \ln(CF - CF^{MIN}) \\ & - \lambda \left(\sum_i P_i C_i + wL + PF CF - y - w LS^{MAX} \right) \end{aligned} \quad [349]$$

Whence, first-order conditions

$$\frac{\partial \Lambda}{\partial C_i} = \frac{\gamma_i}{(C_i - C_i^{MIN})} - \lambda P_i = 0, \text{ that is, } P_i (C_i - C_i^{MIN}) = \frac{\gamma_i}{\lambda} \quad [358]$$

$$\frac{\partial \Lambda}{\partial L} = \frac{\gamma^L}{(L - L^{MIN})} - \lambda w = 0, \text{ that is, } w(L - L^{MIN}) = \frac{\gamma^L}{\lambda} \quad [359]$$

$$\frac{\partial \Lambda}{\partial CF} = \frac{\gamma^F}{(CF - CF^{MIN})} - \lambda \frac{f PAF}{r} = 0, \text{ that is, } \left(\frac{f PAF}{r} \right) (CF - CF^{MIN}) = \frac{\gamma^F}{\lambda} \quad [360]$$

to which is added budget constraint [346].

Condition [359] is equivalent to

$$\frac{1}{\lambda} = \frac{1}{\gamma^L} w(L - L^{MIN}) \quad [361]$$

Substitute [361] into [358] and [360] and there results

$$P_i(C_i - C_i^{MIN}) = \frac{\gamma_i}{\gamma^L} w(L - L^{MIN}) \quad [362]$$

$$\left(\frac{f PAF}{r} \right) CF = \left(\frac{f PAF}{r} \right) CF^{MIN} + \frac{\gamma^F}{\gamma^L} w(L - L^{MIN}) \quad [363]$$

Given [347], the sum of equation [362] over all goods is

$$\sum_i P_i C_i = \sum_i P_i C_i^{MIN} + \frac{1}{\gamma^L} w(L - L^{MIN}) \sum_i \gamma_i = \frac{(1 - \gamma^L - \gamma^F)}{\gamma^L} w(L - L^{MIN}) \quad [364]$$

Substitute [364] and [363] into budget constraint [346] and there follows

$$\begin{aligned} y + w LS^{MAX} &= \sum_i P_i C_i^{MIN} + \frac{(1 - \gamma^L - \gamma^F)}{\gamma^L} w(L - L^{MIN}) + wL \\ &+ \left(\frac{f PAF}{r} \right) CF^{MIN} + \frac{\gamma^F}{\gamma^L} w(L - L^{MIN}) \end{aligned} \quad [365]$$

that is,

$$\begin{aligned} y + w LS^{MAX} - \sum_i P_i C_i^{MIN} - \left(\frac{f PAF}{r} \right) CF^{MIN} \\ = \frac{(1 - \gamma^L - \gamma^F)}{\gamma^L} w(L - L^{MIN}) + wL + \frac{\gamma^F}{\gamma^L} w(L - L^{MIN}) \end{aligned} \quad [366]$$

$$\begin{aligned} y + w LS^{MAX} - \sum_i P_i C_i^{MIN} - \left(\frac{f PAF}{r} \right) CF^{MIN} - wL^{MIN} \\ = \frac{(1 - \gamma^L - \gamma^F)}{\gamma^L} w(L - L^{MIN}) + w(L - L^{MIN}) + \frac{\gamma^F}{\gamma^L} w(L - L^{MIN}) \end{aligned} \quad [367]$$

$$y + w LS^{MAX} - \sum_i P_i C_i^{MIN} - \left(\frac{f PAF}{r} \right) CF^{MIN} - wL^{MIN} = \frac{1}{\gamma^L} w(L - L^{MIN}) \quad [368]$$

Define the **integral** suprenumerary income as it should be when $CF^{MIN} \neq 0$:

$$CSUPINT = y + w LS^{MAX} - \sum_i P_i C_i^{MIN} - w L^{MIN} - PF CF^{MIN} \quad [350]$$

and, given [368],

$$CSUPINT = \frac{1}{\gamma^L} w (L - L^{MIN}) \quad [369]$$

Substitute [369] into [362] and [363] and find the demand equations

$$P_i (C_i - C_i^{MIN}) = \gamma_i CSUPINT \quad [351]$$

$$PF (CF - CF^{MIN}) = \gamma^F CSUPINT \quad [352]$$

Finally, inverting [369],

$$w (L - L^{MIN}) = \gamma^L CSUPINT \quad [353]$$

**PART THREE :
PUBLIC DEBT**

1. Issues related to debt in a CGE

1.1 OBJECTIVE

The objective of this third part is to put forward a few ideas on how to represent the evolution of public debt in a recursive dynamic CGE. To simplify matters, we shall consider public debt to be essentially in the form of bonds. Indeed, bonds are a major part of practically any government's debt, and, moreover, almost all other liabilities share the following characteristics with bonds :

- they are issued at a given date;
- they have a given nominal, or face value;
- they bear interest at a given rate relative to their face value;
- they have an expiry date, at which they are reimbursed by the issuer to the holder.

Treasury bills, although technically different, can nevertheless be represented in the form of bonds. It is therefore as bonds that we propose to represent the whole of public debt. And a specification that would capture these characteristics would be highly desirable.

1.2 BASIC REQUIREMENTS

Speaking of debt in the form of bonds, there are three aspects which call into play the model's « memory », that is, the set of past values which intervene in current period calculations. First, the amount of interest payable depends on the face values and the interest rates of all past issues which have not yet been redeemed. Second, the amount of debt that comes to maturity depends on the face values and maturity dates of all past issues still outstanding. Finally, the level of indebtedness is the result of past issues, that is, of the cumulated deficit of past government expenditures and investment spending.

Of the three aspects mentioned above (interest payments, redemption of mature debt, level of indebtedness), the third is certainly the one that draws most attention. First of all, interest payments are nothing but a consequence of indebtedness : the higher the level of indebtedness, the heavier the burden of interest payments on the government budget. Also, any issuer of securities runs the risk, beyond a certain level of indebtedness, that his/her credit rating fall, which then forces new issues to bear interest at increased rates, and may even close the door to further borrowing.

So it is of utmost importance that be established in the model a relationship between the level of indebtedness and the cost of borrowing. Now, in order to represent the rise in the cost of

borrowing and the erosion of borrowing capacity which results from higher indebtedness, the rate of interest on new issues must depend on the stock of debt. That requires competition to government bonds from of at least one other asset. When government bonds compete with another asset, the greater the stock of outstanding debt, the lower the market valuation of bonds, and the higher the interest rate on new government bond issues.

That modeling strategy implies, first, that there be at least one competing asset, and, second, that the demand for assets reflect the portfolio allocation behavior of asset holders. Moreover, not only current savings, but all of the wealth portfolio must be reallocated in every period. Because, if only current savings are allocated among currently offered new assets, equilibrium prices of new issues are independent of outstanding stocks³⁰.

2. Literature

2.1 THISSEN'S (1999) SURVEY

Thissen (1999) reviews several financial CGE's : Feltenstein (1980, 1984, 1995), Bourguignon, Branson and de Melo (the « Maquette » model; 1989, 1992), Rosensweig and Taylor (1990), Easterly (1990), Fargeix and Sadoulet (1990, 1994) Lewis (1992, 1994), Decaluwé and Nsengiyumva (1994), Souissi and Decaluwé (1997), Yeldan (1997), Vos (1998), and Naastepad (1998)³¹.

Let's first mention that, since the publication of Robinson (1991), all models with financial assets, except Fargeix and Sadoulet (1994), are specified in terms of the stocks of assets, rather than flows (in which case portfolio management is applied incrementally, through the allocation of savings flows). Specification in terms of stocks is indispensable to take into account the consequences of portfolio reallocations, interest payments, and wealth effects.

All models, except Feltenstein's, are described by Thissen as « financial macro CGE models ». As such, they all include money among the financial assets. Moreover, the banking sector is explicitly modeled, as are the relations between the central bank, commercial banks and other agents (borrowers and depositors).

³⁰ The latter approach is Robinson's (1991) or Decaluwé, Martin and Souissi's (1992) « flow of funds » approach.

³¹ Pereira and Shoven (1988) have examined, among other aspects, the treatment of financial assets and of savings in 11 dynamic models. But these are intertemporal dynamic models (at least with respect to consumer behavior), rather than recursive dynamic models of the kind discussed here.

In the present survey, however, we concentrate on how to represent the evolution of public debt in a CGE, rather than on the broader objective of developing a macroeconomic simulation tool. Therefore, we do not think it necessary to introduce money or financial intermediation in the model, neither do we think it relevant to discuss the formation of short-run expectations or the speed of adjustment of goods and services, factor, or financial asset markets. In that respect, a clear distinction has to be made between the objectives of dynamic macroeconomic models and CGE models. Yet, as we have seen, public debt securities must be competing with other assets. That is why we are particularly interested in portfolio management and the list of assets in financial models.

Although it is difficult to generalize, we can say that, in the majority of models, the net wealth of households is adjusted to take into account capital gains and losses. Net wealth is then distributed between physical capital and financial assets, which are treated as imperfect substitutes to one another. The list of assets almost invariably includes money, as we have already pointed out. Some models, like Lewis' (1994), have a single other asset (in that case, interest-bearing bank deposits). The demand for money is then determined by a transactions demand function, and the amount dedicated to the other asset is a residual. More elaborate models generally use a CES utility function to determine portfolio composition; Easterly (1990) uses a logistic function.

2.2 THE « MAQUETTE » MODEL OF BOURGUIGNON, BRANSON AND DE MELO (1989)

The « Maquette » model of Bourguignon, Branson and de Melo (1989) is a macro-simulation model for quantifying the effects of stabilization policies on income and wealth distribution in developing countries. Common shares are absent from the model, because stock markets are virtually non-existent in those countries. In its original version, « Maquette » is specified as a flow-of-funds model. But more recent versions are in terms of stocks.

The model distinguishes five agents and five assets. Government issues bonds which are held by households, banks and the Rest-of-the-World (RoW). Businesses borrow from banks and the RoW. The banking system issues money and receives deposits from households and businesses. In addition to government bonds and bank deposits, households own foreign securities. Finally, the banking system owns foreign currency reserves.

Household wealth is allocated in three stages. First, part is kept in the form of money and bank deposits; the demand for money is a function of income and interest rates. Then, a fraction of the remainder goes to physical capital. The rest is allocated between bonds and foreign

securities. In the latter, third, stage, the allocation mechanism is a constant elasticity function of the following type :

$$\frac{g_a}{1-g_a} = \psi \left(\frac{1+i_a}{1+i_b} \right)^{\varepsilon_a}, \text{ or, equivalently, } g_a = \frac{\psi \left(\frac{1+i_a}{1+i_b} \right)^{\varepsilon_a}}{1 + \psi \left(\frac{1+i_a}{1+i_b} \right)^{\varepsilon_a}} \quad [370]$$

where g_a is the share of asset a , i_a is the rate of return on a , and i_b , the rate of return on competing asset b . Businesses' liabilities are similarly determined, but with a negative value of elasticity.

2.3 THE ROSENSWEIG AND TAYLOR (1990) MODEL

That model has six agents : households, businesses, commercial banks, the central bank, government, and the RoW.

Household wealth in period t consists of the following :

- physical capital (the stock of residential housing and individual unincorporated business capital)
- money
- bank deposits
- shares in businesses
- government bonds

In every period, wealth increases by the amount of current savings. Physical assets are equal to the current value of the capital stock inherited from the preceding period, plus current investments. The latter are a linear function of real household income and of the interest rate. Once the value of physical capital has been subtracted, the remainder of wealth is distributed among the four other assets according to a specification now known as the Rosensweig-Taylor model, which we now describe.

Wealth accumulating households maximize a CES utility function whose arguments are asset returns $z_i V_i$ (note *returns*, and not *rates* of return!) :

$$U = \left[\sum_i A_i (z_i V_i)^\rho \right]^{\frac{1}{\rho}} \quad [371]$$

$$\text{s.t. } \sum_i V_i = W \quad [372]$$

where

- A_i are parameters;
- V_i is the value of asset i in the portfolio;
- $z_i = \frac{r_i}{\bar{r}_i}$ is the ratio of the rate of return of asset i over its « normal » rate of return;
- W is the total value of the portfolio to be allocated.

Parameter ρ is related to the elasticity of substitution as follows : $\rho = \frac{\sigma - 1}{\sigma}$.

Equation [371] reflects the hypothesis that *income flows* from different assets are not perfect substitutes, while the usual assumption is rather that it is the assets themselves that are not perfect substitutes (see below Decaluwé and Souissi, 1994).

The first order optimum conditions lead to demand functions

$$z_i V_i = W \frac{A_i^\sigma z_i^{\sigma-1}}{\zeta}, \text{ où } \zeta = \left[\sum_j A_j^\sigma z_j^{\sigma-1} \right] \quad [373]$$

Rosensweig and Taylor label ζ the « harmonic mean rate of return ». In fact, the ρ -power weighted average of a set of values x_i is defined as

$$M_\rho = \left(\frac{\sum_i w_i (x_i)^\rho}{\sum_i w_i} \right)^{1/\rho} \quad [374]$$

In the particular case where $\rho = -1$ and weights w_i are equal, [374] defines a harmonic mean. It is therefore incorrect to call ζ a harmonic mean, and we will refrain from perpetuating that mistake.

Business investment financing is treated separately for every industry : industry i 's financial needs is the difference between its investment expenditures and its savings. The *net*³²

³² Depreciation does not appear explicitly in the model. It should also be noted that, in Rosensweig and Taylor's notation, capital used in current production bears the time subscript of the preceding period : capital used in period t production is $K_{i,t-1}$.

investment rate of industry i , $\frac{I_{i,t}}{K_{i,t-1}}$, is a linear function of the discrepancy between the rate of return of capital in the industry, net of depreciation, and the rate of interest on bank loans.

In Rosensweig and Taylor, the number of shares issued is a linear function of financial needs in real terms $\frac{DEF_i}{PK_i}$, where DEF_i is the required financing of industry i , and PK_i is the price of capital in industry i . Share prices are determined on the stock market, and, to fulfill their financing requirements, businesses borrow an amount equal to the difference between DEF_i and the proceeds from the new shares issue. The industry (i.e. the representative firm) allocates its liabilities (current and past borrowing) between bank loans and borrowing abroad, minimizing a CES aggregate of interest costs, subject to its financing requirements. The liability portfolio model is similar to the household asset portfolio model, except that businesses minimize the cost of borrowing, while households maximize returns. Naturally, firms' and households' views regarding asset substitutability are not necessarily identical. So the degree of substitutability among assets is determined, not so much by their intrinsic characteristics as by the views of issuers and holders.

Grosso modo, the supply of bank loans is an increasing function of the interest rate, while businesses' demand is a decreasing function of the interest rate: the supply-demand equality condition determines the equilibrium rate of interest.

Government financial needs are the sum of public investment expenditures and acquisition of new share issues by public enterprises, minus government savings (current income and expenditure balance). Foreign loans are exogenous. In Rosensweig and Taylor, the remainder of the government new and old borrowing portfolio is distributed among

- outstanding bonds, the quantity of which is determined by demand, since the rate of return is an exogenously determined policy variable;
- commercial bank loans, a linear function of bank deposits;
- central bank loans, residual.

The amount of outstanding government bonds is determined by the household demand function (the rate of interest is an exogenous policy variable). The amount of shares issued by industries is determined by their financial needs. The amount invested by households in shares depends on the rates of return of shares and bonds. The rate of return on shares is the weighted average

of rates of return on capital. Share prices are consequently equal to the ratio of the aggregate value of household stock market holdings over the number of shares issued by all industries.

2.4 COLLANGE'S (1993) IVORY COAST MODEL

Collange's (1993) model combines the Rosensweig-Taylor portfolio management model and a liability management model à la Bourguignon *et al.* Collange distinguishes six agents : households, businesses, commercial banks, the central bank, government, and the RoW.

Household wealth consists of money, bank deposits and foreign securities, and it increases each period by the amount of savings. The quantity of money held by households is determined by a demand function of the form $\phi Y r_{hm}^\varepsilon$, where ϕ is a constant, Y is income, and r_{hm} , the so-called « harmonic mean return » of deposits and foreign securities, with elasticity ε negative. The remainder of wealth is allocated between deposits and foreign securities à la Rosensweig-Taylor.

After taking into account certain transfers and exogenous flows, businesses' financing needs are equal to the difference between their investment demand and their savings. Their complementary sources of financing are domestic and foreign loans; allocation between the two is given by a distribution function similar to that of Bourguignon *et al.*

Public sector financing needs are equal to the difference between public investment and the sum of current government savings and (exogenous) transfers received by businesses³³. These needs are met by central bank advances, and, either payment delays, or foreign capital inflows (one of the two is exogenous).

Commercial bank financing needs are the difference between, on one hand, credit to business and reserves deposited with the central bank, and, on the other hand, household deposits and the banks' own savings. The rest is made up for by foreign loans and central bank refinancing, with a Bourguignon *et al.* distribution function.

As for the central bank, either, if the nominal exchange rate is exogenous, it balances its account with currency reserve adjustments, or equilibrium is achieved thanks to nominal exchange rate variations.

Interest rates are all tied to foreign rates, which are exogenous, and to the preceding year's refinancing rate. In the model's equations, foreign interest rates i_f always appear as $(1+i_f)(1+\hat{\epsilon})$,

where $\hat{\epsilon}$ is identified as the « expected rate of exchange »; more likely, it is the expected proportional change in the exchange rate. Insofar as that expected variation depends on the current observed variations, and if financial equilibrium is achieved through exchange rate adjustment, it is the only channel through which interest rates move in response to asset market tensions.

2.5 DECALUWÉ-SOUISSI'S (1994) AND SOUISSI'S (1994) MODEL

We agree with Souissi's (1994) criticism of asset demand in the Rosensweig-Taylor model. According to Souissi, the portfolio manager can acquire a unit of asset i at the beginning of period t , for a price equal to q_i , and this will yield an investment income of $r_i q_i$ at the end of period t (at the beginning of $t+1$), resulting in a capitalized value of $\xi_i = (1+r_i) q_i$ at the end of period t (at the beginning of $t+1$).

In each period t , every portfolio manager maximizes the capitalized value of his/her wealth

$$\text{MAX}_{a_i} VC = \sum_i \xi_i a_i, \text{ where } \xi_i = (1+r_i)q_i \quad [375]$$

subject to³⁴

$$W = A_w \left[\sum_i \delta_i a_i^\beta \right]^{\frac{1}{\beta}} \quad [376]$$

with transformation elasticity

$$\tau = \frac{1}{1-\beta} \quad (\beta > 1) \quad [377]$$

Clearly, the form of the utility function implies that all of the portfolio be reallocated in every period. It follows that wealth W consists of the value of assets owned at the end of period $t-1$ and of current savings.

Taking into account the household wealth accounting identity

$$\sum_i q_i a_i = W \quad [378]$$

³³ Collange doesn't specify the exact nature of these transfers.

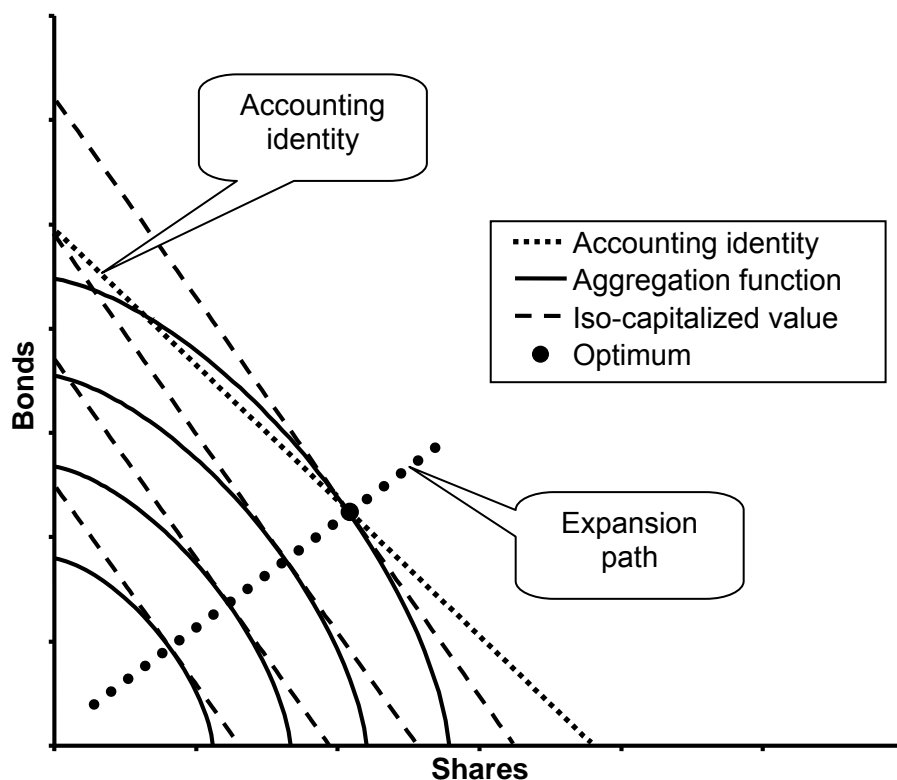
³⁴ In his statement of the model, Souissi describes that constraint as the wealth constraint. But A_w in [376] is not a parameter : it is a variable whose value depends on wealth accounting identity [378]. That is why we prefer to call [376] a diversification constraint. That aspect of the Decaluwé-Souissi model is discussed in more detail in Lemelin (2005), especially in Annex 2.

leads to demand functions

$$q_i a_i = W \frac{\delta_i^\tau q_i \xi_i^{-\tau}}{\sum_j \delta_j^\tau q_j \xi_j^{-\tau}} \quad [379]$$

The Decaluwé-Souissi portfolio allocation model is illustrated in the following diagram.

Figure 1 – Household portfolio allocation



Portfolio allocation equilibrium is located at the intersection of the expansion path and the wealth accounting identity constraint. The expansion path consists of the set of optimal asset combinations, for given return rates and different levels of wealth; for any optimal asset combination, the marginal rate of transformation of the diversification constraint is equal to the slope of the iso-capitalized value line (whose equation is given by [375], with a constant value for VC).

2.6 THE LEMELIN (2005, 2007) MODEL

Lemelin (2005, 2007) proposes a model which can account for interest payments, debt redemption at maturity, and the level of indebtedness, while maintaining acceptable model memory requirements.

Lemelin's (2005, 2007) formulation does not pretend to be operational. The stated objective is rather to present the general principle of the proposed specification in a minimalist form. It is a model without money, and consequently, it cannot be considered to be a financial model. Moreover, there is no financial intermediation in the model. And the range of assets is as simplified as it can be.

Lemelin's (2005) minimalist model has three agents : households, businesses and government :

- Government issues bonds to finance its current deficit and public investment.
- Businesses issue shares to finance their investment expenditures.
- Households own a portfolio of both assets.

Bonds compete with shares, so that the rate of return demanded on new bond issues rises as the amount of outstanding bonds increases relative to the stock of outstanding shares.

By imposing somewhat restrictive assumptions relative to the maturity structure of public debt, Lemelin (2005) claims to have achieved a reasonable compromise between a realistic representation of the evolution of public debt and the weight of past variable values to be kept in the model's memory.

In the proposed model, government redeems bonds that have come to maturity and pays interest on the outstanding debt. The price of bonds issued in different periods with different maturities is consistent with an arbitrage equilibrium. The supply of new bonds and new shares is determined by government and business financing requirements. Asset demand reflects rational household portfolio management behavior, following the Decaluwé-Souissi approach. However, Lemelin modifies the Decaluwé-Souissi model in one respect : the diversification constraint is stated in terms of the *value* of different assets in the portfolio. The diversification constraint becomes

$$W = A_w \left[\sum_i \delta_i (q_i a_i)^\beta \right]^{\frac{1}{\beta}} \quad [379.1]$$

The whole model can then be reformulated in terms of asset values. Let

$$b_i = q_i a_i \quad [379.2]$$

The Decaluwé-Souissi objective function can be written

$$\underset{b_i}{MAX} VC = \sum_i (1 + r_i) b_i \quad [379.3]$$

subject to

$$W = A_w \left[\sum_i \delta_i b_i^\beta \right]^{\frac{1}{\beta}} \quad [379.4]$$

and the household wealth accounting identity

$$\sum_i b_i = W \quad [379.5]$$

This leads to asset demand functions of the form

$$b_i = W \frac{\delta_i^\tau (1 + r_i)^{-\tau}}{\sum_j \delta_j^\tau (1 + r_j)^{-\tau}} \quad [379.6]$$

The model is developed in two versions : a basic model, and a model which, while it doesn't have money, includes a mechanism whereby the real value of outstanding bonds is eroded by inflation.

The feasibility of the modeling principle presented in Lemelin (2005) is demonstrated in Lemelin (2007), by means of the EXTER-Debt model, a small-scale recursive dynamic CGE model based on fictitious data.

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COMPENDIUM OF MATHEMATICAL EXPRESSIONS

$$K_{i,t+1} = I_{i,t} + (1 - \delta_i)K_{i,t} \quad [001]$$

$$p_t F(K_t, L_t) - w_t L_t - q_t I_t \quad [002]$$

$$\dot{K}_t = I_t - \delta K_t \quad [003]$$

$$MAX V = \int_0^{\infty} e^{-rt} [p_t F(K_t, L_t) - w_t L_t - q_t I_t] dt \quad [004]$$

$$K_0 = \bar{K}_0 \quad [005]$$

$$p_t \frac{\partial F}{\partial L_t} = w_t \quad [006]$$

$$p_t \frac{\partial F}{\partial K_t} = q_t \left[r + \delta - \frac{\dot{q}_t}{q_t} \right] \quad [007]$$

$$\pi_t = \frac{\dot{q}_t}{q_t} \quad [008]$$

$$u_t = q_t (r + \delta - \pi_t) \quad [009]$$

$$MAX [p_t F(K_t, L_t) - w_t L_t - u_t K_t] \quad [010]$$

$$K_t = \bar{K}_0 + \int_0^t \dot{K}_\tau d\tau = \bar{K}_0 + \int_0^t (I_\tau - \delta K_\tau) d\tau \quad [011]$$

$$p_t \frac{\partial F}{\partial K_t} = u_t \quad [012]$$

$$\Delta_t K = K_{t+1} - K_t = I_t - \delta K_t \quad [013]$$

$$I_t = K_{t+1} - K_t + \delta K_t = K_{t+1} - (1 - \delta)K_t \quad [014]$$

$$MAX V = \sum_{t=0}^{\infty} \frac{1}{(1+r)^t} [p_t F(K_t, L_t) - w_t L_t - q_t (K_{t+1} - (1 - \delta)K_t)] \quad [015]$$

$$p_t \frac{\partial F}{\partial L_t} = w \quad [016]$$

$$p_t \frac{\partial F}{\partial K_t} = (1+r)q_{t-1} - (1 - \delta)q_t \quad [017]$$

$$\rho_t \frac{\partial F}{\partial K_t} = r q_{t-1} + \delta q_t - (q_t - q_{t-1}) \quad [018]$$

$$\pi_t = \frac{(q_t - q_{t-1})}{q_{t-1}} \quad [019]$$

$$(q_t - q_{t-1}) = \frac{(q_t - q_{t-1})}{q_{t-1}} q_{t-1} = \pi_t q_{t-1} \quad [020]$$

$$\rho_t \frac{\partial F}{\partial K_t} = r q_{t-1} + \delta q_t - \pi_t q_{t-1} = (r - \pi_t) q_{t-1} + \delta q_t \quad [021]$$

$$u_t = (r - \pi_t) q_{t-1} + \delta q_t \quad [022]$$

$$\rho \Delta K = \sum_{t=1}^{\infty} \left(\frac{1}{1+\rho} \right)^t E(t) \quad [023]$$

$$\rho \Delta K = \sum_{t=1}^{\infty} \left(\frac{1}{1+\rho} \right)^t E(t) = \sum_{t=1}^{\infty} \left(\frac{1}{1+\rho} \right)^t \bar{E} = \frac{1}{\rho} \bar{E} \quad [024]$$

$$MV = \sum_{t=1}^{\infty} \left(\frac{1}{1+r_K} \right)^t E(t) \quad [025]$$

$$MV = \sum_{t=1}^{\infty} \left(\frac{1}{1+r_K} \right)^t E(t) = \sum_{t=1}^{\infty} \left(\frac{1}{1+r_K} \right)^t \bar{E} = \frac{1}{r_K} \bar{E} \quad [026]$$

$$q = \frac{MV}{\rho \Delta K} \quad [027]$$

$$q = \frac{MV}{\rho \Delta K} = \frac{\left(\frac{\bar{E}}{r_K} \right)}{\left(\frac{\bar{E}}{\rho} \right)} = \frac{\left(\frac{1}{r_K} \right)}{\left(\frac{1}{\rho} \right)} = \frac{\rho}{r_K} \quad [028]$$

$$q_t K_{t+1} = \frac{1}{(1+r)} \left[(1-\delta) q_{t+1} K_{t+1} + \rho_{t+1} \frac{\partial F}{\partial K_{t+1}} K_{t+1} \right] \quad [029]$$

$$(1-\delta) K_t = K_{t+1} - I_t \quad [030]$$

$$q_t K_{t+1} = \frac{1}{(1+r)} \left(\rho_{t+1} \frac{\partial F}{\partial K_{t+1}} K_{t+1} - q_{t+1} I_{t+1} + q_{t+1} K_{t+2} \right) \quad [031]$$

$$q_t K_{t+1} = \sum_{s=1}^{\infty} \frac{1}{(1+r)^s} \left[\rho_{t+s} \frac{\partial F}{\partial K_{t+s}} K_{t+s} - q_{t+s} I_{t+s} \right] + \lim_{s \rightarrow \infty} \left(\frac{1}{(1+r)^s} q_{t+s} K_{t+s+1} \right) \quad [032]$$

$$\lim_{s \rightarrow \infty} \frac{1}{(1+r)^s} q_{t+s} K_{t+s+1} = 0 \quad [033]$$

$$\text{MAX } V = \sum_{t=0}^T \frac{1}{(1+r)^t} [p_t F(K_t, L_t) - w_t L_t - q_t (K_{t+1} - (1-\delta)K_t)] \quad [034]$$

$$\text{MAX } V = \sum_{t=0}^T \frac{1}{(1+r)^t} [p_t F(K_t, L_t) - w_t L_t] - \sum_{t=0}^T \frac{1}{(1+r)^t} q_t K_{t+1} + \sum_{t=0}^T \frac{1}{(1+r)^t} q_t (1-\delta)K_t \quad [035]$$

$$\frac{1}{(1+r)^T} q_T K_{T+1} > 0 \quad [036]$$

$$\frac{1}{(1+r)^T} q_T K_{T+1} = 0 \quad [037]$$

$$\text{Lim}_{T \rightarrow \infty} \frac{1}{(1+r)^T} q_T K_{T+1} = 0 \quad [038]$$

$$q_t K_{t+1} = \sum_{s=1}^{\infty} \frac{1}{(1+r)^s} \left[p_{t+s} \frac{\partial F}{\partial K_{t+s}} K_{t+s} - q_{t+s} l_{t+s} \right] \quad [039]$$

$$F(K_t, L_t) = \frac{\partial F}{\partial K_t} K_t + \frac{\partial F}{\partial L_t} L_t \quad [040]$$

$$p_t \frac{\partial F}{\partial K_t} K_t = p_t F(K_t, L_t) - w_t L_t \quad [041]$$

$$q_t K_{t+1} = \sum_{s=1}^{\infty} \frac{1}{(1+r)^s} [p_{t+s} F(K_{t+s}, L_{t+s}) - w_{t+s} L_{t+s} - q_{t+s} l_{t+s}] \quad [042]$$

$$\frac{\sum_{s=1}^{\infty} \frac{1}{(1+r)^s} [p_{t+s} F(K_{t+s}, L_{t+s}) - w_{t+s} L_{t+s} - q_{t+s} l_{t+s}]}{q_t K_{t+1}} = 1 \quad [043]$$

$$q_t = \frac{1}{(1+r)} \left\{ \begin{array}{l} \frac{(1-\delta)^3}{(1+r)^2} q_{t+3} + \frac{(1-\delta)^2}{(1+r)^2} p_{t+3} \frac{\partial F}{\partial K_{t+3}} \\ + \frac{(1-\delta)}{(1+r)} p_{t+2} \frac{\partial F}{\partial K_{t+2}} \\ + p_{t+1} \frac{\partial F}{\partial K_{t+1}} \end{array} \right\} \quad [044]$$

$$q_t = \left(\frac{1-\delta}{1+r} \right)^\theta q_\theta + \frac{1}{(1+r)} \sum_{s=1}^{\theta} \left(\frac{1-\delta}{1+r} \right)^{s-1} p_{t+s} \frac{\partial F}{\partial K_{t+s}} \quad [045]$$

$$q_t = \lim_{\theta \rightarrow \infty} \left(\frac{1-\delta}{1+r} \right)^\theta q_\theta + \frac{1}{(1+r)} \sum_{s=1}^{\infty} \left(\frac{1-\delta}{1+r} \right)^{s-1} p_{t+s} \frac{\partial F}{\partial K_{t+s}} \quad [046]$$

$$\frac{1+r}{1-\delta} - 1 = \frac{r+\delta}{1-\delta} \quad [047]$$

$$\lim_{\theta \rightarrow \infty} \left(\frac{1-\delta}{1+r} \right)^\theta q_\theta > 0 \quad [048]$$

$$q_t > \frac{1}{(1+r)} \sum_{s=1}^{\infty} \left(\frac{1-\delta}{1+r} \right)^{s-1} p_{t+s} \frac{\partial F}{\partial K_{t+s}} \quad [049]$$

$$\lim_{t \rightarrow \infty} \left(\frac{1-\delta}{1+r} \right)^t q_t = 0 \quad [050]$$

$$q_t = \frac{1}{(1+r)} \sum_{s=1}^{\infty} \left(\frac{1-\delta}{1+r} \right)^{s-1} p_{t+s} \frac{\partial F}{\partial K_{t+s}} \quad [051]$$

$$R_t = p_t \frac{\partial F}{\partial K_t} \quad [052]$$

$$q_t = \frac{1}{(1+r)} \sum_{s=0}^{\infty} \left(\frac{1-\delta}{1+r} \right)^s R_{t+s+1} \quad [053]$$

$$\tilde{R}_{t+s} = R_t, \forall s \geq 0 \quad [054]$$

$$R_t = (r+\delta)q_t \quad [055]$$

$$\bar{u}_t = (r+\delta)q_t \quad [056]$$

$$\pi_t = \frac{(q_t - q_{t-1})}{q_{t-1}} = 0, \text{ which implies } q_t = q_{t-1} \quad [057]$$

$$R_t = (r+\delta)q_t = \bar{u}_t \quad [058]$$

$$C'(l_t) > \text{ or } < 0 \Leftrightarrow l_t > \text{ or } < 0 \quad [059]$$

$$C(0) = 0 \quad [060]$$

$$C''(l_t) > 0 \quad [061]$$

$$C(l_t) = q_t \frac{\gamma}{2} l_t^2 \quad [062]$$

$$\text{MAX } V = \sum_{t=0}^{\infty} \frac{1}{(1+r)^t} [p_t F(K_t, L_t) - w_t L_t - q_t l_t - C(l_t)] \quad [063]$$

$$p_t \frac{\partial F}{\partial K_t} - (1+r)q_{t-1}(1+\gamma l_{t-1}) + q_t(1+\gamma l_t)(1-\delta) = 0 \quad [064]$$

$$Q_t = \frac{\partial}{\partial l_t} \left[q_t l_t \left(1 + \frac{\gamma}{2} l_t \right) \right] = q_t (1 + \gamma l_t) \quad [065]$$

$$p_t \frac{\partial F}{\partial K_t} = (1+r)Q_{t-1} - (1-\delta)Q_t \quad [066]$$

$$\Pi_t = \frac{(Q_t - Q_{t-1})}{Q_{t-1}} \quad [067]$$

$$(Q_t - Q_{t-1}) = \frac{(Q_t - Q_{t-1})}{Q_{t-1}} Q_{t-1} = \Pi_t Q_{t-1} \quad [068]$$

$$p_t \frac{\partial F}{\partial K_t} = r Q_{t-1} + \delta Q_t - (Q_t - Q_{t-1}) \quad [069]$$

$$p_t \frac{\partial F}{\partial K_t} = (r - \Pi_t) Q_{t-1} + \delta Q_t \quad [070]$$

$$U_t = (r - \Pi_t) Q_{t-1} + \delta Q_t \quad [071]$$

$$p_t \frac{\partial F}{\partial K_t} = U_t \quad [072]$$

$$Q_t K_{t+1} = \frac{1}{(1+r)} [(1-\delta) Q_{t+1} K_{t+1} + R_{t+1} K_{t+1}] \quad [073]$$

$$Q_t K_{t+1} = \frac{1}{(1+r)} (R_{t+1} K_{t+1} - Q_{t+1} l_{t+1} + Q_{t+1} K_{t+2}) \quad [074]$$

$$Q_t K_{t+1} = \frac{1}{(1+r)} \left[\frac{R_{t+1} K_{t+1} - Q_{t+1} l_{t+1}}{(1+r)} + \frac{1}{(1+r)} (R_{t+2} K_{t+2} - Q_{t+2} l_{t+2} + Q_{t+2} K_{t+3}) \right] \quad [075]$$

$$Q_t K_{t+1} = \sum_{s=1}^{\infty} \frac{1}{(1+r)^s} [R_{t+s} K_{t+s} - Q_{t+s} l_{t+s}] + \lim_{s \rightarrow \infty} \left(\frac{1}{(1+r)^s} Q_{t+s} K_{t+s+1} \right) \quad [076]$$

$$\lim_{s \rightarrow \infty} \left(\frac{1}{(1+r)^s} Q_{t+s} K_{t+s+1} \right) = 0 \quad [077]$$

$$Q_t K_{t+1} = \sum_{s=1}^{\infty} \frac{1}{(1+r)^s} [R_{t+s} K_{t+s} - Q_{t+s} l_{t+s}] \quad [078]$$

$$R_t K_t = p_t F(K_t, L_t) - w_t L_t \quad [079]$$

$$Q_t K_{t+1} = \sum_{s=1}^{\infty} \frac{1}{(1+r)^s} [p_{t+s} F(K_{t+s}, L_{t+s}) - w_{t+s} L_{t+s} - Q_{t+s} I_{t+s}] \quad [080]$$

$$\frac{\sum_{s=1}^{\infty} \frac{1}{(1+r)^s} \left[p_{t+s} F(K_{t+s}, L_{t+s}) - w_{t+s} L_{t+s} \right]}{Q_t K_{t+1}} = 1 + \frac{\sum_{s=1}^{\infty} \frac{1}{(1+r)^s} \left[q_{t+s} I_{t+s} \left(\frac{\gamma}{2} I_{t+s} \right) \right]}{Q_t K_{t+1}} \quad [081]$$

$$C(I_t, K_t) = q_t \frac{\gamma}{2} \frac{I_t^2}{K_t} \quad [082]$$

$$\frac{\sum_{s=1}^{\infty} \frac{1}{(1+r)^s} \left[p_{t+s} F(K_{t+s}, L_{t+s}) - w_{t+s} L_{t+s} - q_{t+s} I_{t+s} \left(1 + \frac{\gamma}{2} \frac{I_{t+s}}{K_{t+s}} \right) \right]}{Q_t K_{t+1}} = 1 \quad [083]$$

$$Q_t = \frac{(1+r)}{(1-\delta)} Q_{t-1} - \frac{1}{(1-\delta)} p_t \frac{\partial F}{\partial K_t} \quad [084]$$

$$Q_t = \frac{1}{(1+r)} \left\{ (1-\delta) Q_{t+1} + p_{t+1} \frac{\partial F}{\partial K_{t+1}} \right\} \quad [085]$$

$$Q_t = \frac{1}{(1+r)} \left\{ \begin{aligned} & \frac{(1-\delta)^3}{(1+r)^2} Q_{t+3} + \frac{(1-\delta)^2}{(1+r)^2} p_{t+3} \frac{\partial F}{\partial K_{t+3}} \\ & + \frac{(1-\delta)}{(1+r)} p_{t+2} \frac{\partial F}{\partial K_{t+2}} \\ & + p_{t+1} \frac{\partial F}{\partial K_{t+1}} \end{aligned} \right\} \quad [086]$$

$$\lim_{t \rightarrow \infty} \left(\frac{1-\delta}{1+r} \right)^t Q_t = 0 \quad [087]$$

$$Q_t = \frac{1}{(1+r)} \sum_{s=1}^{\infty} \left(\frac{1-\delta}{1+r} \right)^{s-1} p_{t+s} \frac{\partial F}{\partial K_{t+s}} \quad [088]$$

$$Q_t = \frac{1}{(1+r)} \sum_{s=1}^{\infty} \left(\frac{1-\delta}{1+r} \right)^{s-1} R_{t+s} \quad [089]$$

$$Q_t = \frac{1}{(1+r)} \sum_{s=0}^{\infty} \left(\frac{1-\delta}{1+r} \right)^s R_{t+s+1} \quad [090]$$

$$I_t = \frac{1}{\gamma} \left(\frac{Q_t}{q_t} - 1 \right) \quad [091]$$

$$Q_t = \frac{1}{(1+r)} \sum_{s=1}^{\infty} \left(\frac{1-\delta}{1+r} \right)^{s-1} \tilde{R}_{t+s} \quad [092]$$

$$Q_t = \frac{1}{(1+r)} \sum_{s=1}^{\infty} \left(\frac{1-\delta}{1+r} \right)^{s-1} R_t \quad [093]$$

$$q_t = \frac{1}{(1+r)} \left(\frac{1}{1 - \frac{1-\delta}{1+r}} \right) R_t \quad [094]$$

$$q_t = \frac{1}{(1+r)} \frac{1}{\left(\frac{r+\delta}{1+r} \right)} R_t \quad [095]$$

$$Q_t = \frac{1}{(r+\delta)} R_t \quad [097]$$

$$R_t = (r+\delta) Q_t \quad [098]$$

$$I_t = \frac{1}{\gamma} \left(\frac{1}{q_t} \frac{1}{(r+\delta)} R_t - 1 \right) \quad [099]$$

$$\tilde{Q}_{t+s} = \frac{1}{(r+\delta)} \tilde{R}_{t+s} = \frac{1}{(r+\delta)} R_t \quad [100]$$

$$\tilde{I}_{t+s} = \frac{1}{\gamma} \left(\frac{1}{\tilde{q}_{t+s}} \frac{1}{(r+\delta)} \tilde{R}_{t+s} - 1 \right) = \frac{1}{\gamma} \left(\frac{1}{q_t} \frac{1}{(r+\delta)} R_t - 1 \right) = I_t \quad [101]$$

$$\tilde{K}_{t+s} = (1-\delta) \tilde{K}_{t+s-1} + \tilde{I}_{t+s-1} \quad [102]$$

$$\tilde{K}_{t+s} = (1-\delta) \left[(1-\delta) \tilde{K}_{t+s-2} + \tilde{I}_{t+s-2} \right] + \tilde{I}_{t+s-1} \quad [103]$$

$$\tilde{K}_{t+s} = (1-\delta)^s K_t + I_t \left(\sum_{\theta=1}^s (1-\delta)^{\theta-1} \right) \quad [104]$$

$$\frac{\partial F}{\partial K_t} = \frac{F(K_t, L_t)}{K_t} - \frac{L_t}{K_t} \frac{\partial F}{\partial L_t} \quad [105]$$

$$\frac{\partial F}{\partial K_t} = F \left(1, \frac{L_t}{K_t} \right) - \frac{L_t}{K_t} \frac{\partial F}{\partial L_t} \quad [106]$$

$$\frac{\partial F}{\partial K_t} = F \left(1, \frac{L_t}{K_t} \right) - \frac{L_t}{K_t} \frac{w_t}{p_t} \quad [107]$$

$$p_t \frac{\partial F}{\partial K_t} = (r + \delta) Q_t \quad [108]$$

$$F(K_t, L_t) = \frac{(r + \delta) Q_t}{p_t} K_t + \frac{w_t}{p_t} L_t \quad [109]$$

$$F\left(1, \frac{L_t}{K_t}\right) = \frac{(r + \delta) Q_t}{p_t} + \frac{w_t}{p_t} \frac{L_t}{K_t} \quad [110]$$

$$I_t = \frac{1}{\gamma} \left(\frac{R_t}{\bar{u}_t} - 1 \right) \quad [111]$$

$$I_t = a \left\{ \frac{p^n MP_k U}{q(\delta + J^F)} - 1 \right\} = a \left\{ \frac{B}{C} - 1 \right\} \geq 0 \quad [112]$$

$$\frac{I_t}{K_t} = q\gamma_1 \left[\left(\frac{B}{C} \right)^2 + \gamma_2 \left(\frac{B}{C} \right) \right] \quad [113]$$

$$\frac{I_t}{K_t} = \frac{1}{\gamma} \left(\frac{R_t}{\bar{u}_t} - 1 \right) \quad [114]$$

$$\frac{INV_{it}}{K_{it}} = A_i \left(\frac{KINC_{it}}{PK_{it} K_{it} r_t} \right)^{\beta_i} \quad [115]$$

$$\sum_{s=1}^{\infty} \frac{1}{(1+r)^s} [p_t F(K_t, L_t) - w_t L_t] = \frac{1}{r} [p_t F(K_t, L_t) - w_t L_t] \quad [116]$$

$$\frac{INV_{it}}{K_{it}} = A_i \quad [117]$$

$$\frac{INV_{it}}{K_{it}} = A_i \left(\frac{KINC_{it}}{PK_{it} K_{it} (r_t + \delta)} \right)^{\beta_i} \quad [118]$$

$$\frac{I}{K} = z \left[\frac{profit}{KPK \left(i^* + \delta - \frac{\Delta PK}{PK} \right)} \right]^{\sigma} \quad [119]$$

$$\frac{I_{it}}{K_{it}} = B_i \left(\frac{KINC_{it} (1 + \pi_t)}{PK_{it} K_{it} (1 + rd_t)} \right)^{\varepsilon_i} \quad [120]$$

$$\sum_{s=1}^{\infty} \left(\frac{1+\pi_t}{1+rd_t} \right)^{t+s} KING_t = \frac{1+\pi_t}{rd_t - \pi_t} KING_t \quad [121]$$

$$\frac{I_i}{K_i} = B_i \left(\frac{RK_i}{PK_i} \right)^{\sigma_1} \left(\frac{J_e}{1+PINFL} \right)^{\sigma_2} \left(\frac{Autofin}{Autofin_0} \right)^{\sigma_3} \quad [122]$$

$$\sum_{t=0}^{\infty} \left(\frac{rs_i IND_i}{(1+TIN)^t} \right) = \frac{rs_i IND_i}{TIN} \quad [123]$$

$$MAX \sum_i \left(\frac{rs_i}{TIN} - PK_i \right) IND_i \quad [124]$$

$$\text{s.c. } \sum_i PK_i IND_i \leq IT \quad [125]$$

$$\Lambda = \sum_i \left(\frac{rs_i}{TIN} - PK_i \right) IND_i + \lambda \left(IT - \sum_i PK_i IND_i \right) \quad [126]$$

$$\frac{\partial \Lambda}{\partial IND_i} = \left(\frac{rs_i}{TIN} - PK_i \right) - \lambda PK_i \leq 0 \quad [127]$$

$$IND_i \geq 0 \text{ (non-negativity constraint)} \quad [128]$$

$$IND_i \left[\left(\frac{rs_i}{TIN} - PK_i \right) - \lambda PK_i \right] = 0 \text{ (orthogonality constraint)} \quad [129]$$

$$\left(IT - \sum_i PK_i IND_i \right) \geq 0 \quad [130]$$

$$\lambda \geq 0 \text{ (non-negativity constraint)} \quad [131]$$

$$\lambda \left(IT - \sum_i PK_i IND_i \right) = 0 \text{ (orthogonality constraint)} \quad [132]$$

$$\frac{rs_i}{TIN} - (1+\lambda) PK_i \leq 0 \quad [133]$$

$$\frac{rs_i}{TIN} \leq (1+\lambda) PK_i \quad [134]$$

$$\frac{\left(\frac{rs_i}{TIN} \right)}{PK_i} \leq (1+\lambda) \quad [135]$$

$$\frac{\left(\frac{rs_i}{TIN} \right)}{PK_i} < (1+\lambda) \text{ (strict inequality)} \quad [136]$$

$$v_i = \left(\frac{rs_i}{TIN} - PK_i \right) \quad [137]$$

$$U_{in} = \beta_i v_i + \varepsilon_{in} \quad [138]$$

$$\text{Prob}[E \leq \varepsilon] = F(\varepsilon) = \exp[-e^{-\mu(\varepsilon - \eta)}] \quad [139]$$

$$\text{Pr}_n(i) = \frac{\exp(\beta_i v_i)}{\sum_j \exp(\beta_j v_j)} \quad [140]$$

$$K^S = \left[\sum_i \left(\gamma_i^k \right)^{\omega^K} (KS_i)^{\frac{1+\omega^K}{\omega^K}} \right]^{\frac{\omega^K}{1+\omega^K}} \quad [141]$$

$$KS_i^S = \gamma_i^k \left(\frac{R_i}{TR} \right)^{\omega^K} K^S \text{ si } 0 \leq \omega^K < \infty \quad [142]$$

$$R_i = TR \text{ si } \omega^K = \infty \quad [143]$$

$$TR = \left[\sum_i \gamma_i^k (R_i)^{1+\omega^K} \right]^{\frac{1}{1+\omega^K}} \text{ if } 0 \leq \omega^K < \infty \quad [144]$$

$$\sum_i K_i^d = K^S \text{ if } \omega^K = \infty \quad [145]$$

$$\frac{\frac{R_{i,t}^{old}}{R_{i,t}^{new}}}{\frac{R_{i,t-1}^{old}}{R_{i,t-1}^{new}}} \quad [146]$$

$$F = \frac{Gm_1 m_2}{d^2} \quad [147]$$

$$N_{od} = \frac{G_{od} O_o D_d}{f(d_{od})} \quad [148]$$

$$\sum_o N_{od} = D_d \sum_o \frac{G_{od} O_o}{f(d_{od})} = D_d \text{ for any destination } d \quad [149]$$

$$\sum_d N_{od} = O_o \sum_d \frac{G_{od} D_d}{f(d_{od})} = O_o \text{ for any origin } o \quad [150]$$

$$PK_{k,t} IND_{k,i,rg,t} = \frac{G_{k,i,rg,t} D_{k,i,rg,t}}{f(d_{k,i,rg,t})} \quad [151]$$

$$\sum_k \sum_i \sum_{rg} PK_{k,t} IND_{k,i,rg,t} = \sum_k \sum_i \sum_{rg} \frac{G_{k,i,rg,t} D_{k,i,rg,t}}{f(d_{k,i,rg,t})} = IT_t \quad [152]$$

$$f(d_{k,i,rg,t}) = e^{-\alpha rs_{k,i,rg,t}}, \text{ where } \alpha \text{ is a free parameter.} \quad [153]$$

$$PK_{k,t} IND_{k,i,rg,t} = G_{k,i,rg,t} e^{\alpha rs_{k,i,rg,t}} PK_{k,t} KS_{k,i,rg,t} \quad [154]$$

$$\sum_k \sum_i \sum_{rg} PK_{k,t} IND_{k,i,rg,t} = \sum_k \sum_i \sum_{rg} \left(G_{k,i,rg,t} e^{\alpha rs_{k,i,rg,t}} PK_{k,t} KS_{k,i,rg,t} \right) = IT_t \quad [155]$$

$$G_{k,i,rg,t} = \frac{A_{k,i,rg} IT_t}{\sum_{kj} \sum_j \sum_{rgj} \left(e^{\alpha rs_{kj,j,rgj,t}} A_{kj,j,rgj} PK_{kj,t} KS_{kj,j,rgj,t} \right)} \quad [156]$$

$$PK_{k,0} IND_{k,i,rg,0} = G_{k,i,rg,0} e^{\alpha rs_{k,i,rg,0}} PK_{k,0} KS_{k,i,rg,0} \quad [157]$$

$$PK_{k,0} IND_{k,i,rg,0} = \frac{A_{k,i,rg} IT_0 e^{\alpha rs_{k,i,rg,0}} PK_{k,0} KS_{k,i,rg,0}}{\sum_{kj} \sum_j \sum_{rgj} \left(e^{\alpha rs_{kj,j,rgj,0}} A_{kj,j,rgj,0} PK_{kj,0} KS_{kj,j,rgj,0} \right)} \quad [158]$$

$$\begin{aligned} & \frac{A_{k,i,rg} IT_0 e^{\alpha rs_{k,i,rg,0}} PK_{k,0} KS_{k,i,rg,0}}{\sum_{kj} \sum_j \sum_{rgj} \left(e^{\alpha rs_{kj,j,rgj,0}} A_{kj,j,rgj,0} PK_{kj,0} KS_{kj,j,rgj,0} \right)} \\ &= \frac{\lambda A_{k,i,rg} IT_0 e^{\alpha rs_{k,i,rg,0}} PK_{k,0} KS_{k,i,rg,0}}{\lambda \sum_{kj} \sum_j \sum_{rgj} \left(e^{\alpha rs_{kj,j,rgj,0}} A_{kj,j,rgj,0} PK_{kj,0} KS_{kj,j,rgj,0} \right)} \\ &= \frac{\lambda A_{k,i,rg} IT_0 e^{\alpha rs_{k,i,rg,0}} PK_{k,0} KS_{k,i,rg,0}}{\sum_{kj} \sum_j \sum_{rgj} \left(e^{\alpha rs_{kj,j,rgj,0}} \lambda A_{kj,j,rgj,0} PK_{kj,0} KS_{kj,j,rgj,0} \right)} \end{aligned} \quad [159]$$

$$\sum_{kj} \sum_j \sum_{rgj} \left(e^{\alpha rs_{kj,j,rgj,0}} \lambda A_{kj,j,rgj,0} PK_{kj,0} KS_{kj,j,rgj,0} \right) = 1 \quad [160]$$

$$PK_{k,i,rg,0} IND_{k,i,rg,0} = A_{k,i,rg} IT_0 e^{\alpha rs_{k,i,rg,0}} PK_{k,i,rg,0} KS_{k,i,rg,0} \quad [161]$$

$$A_{k,i,rg} = \frac{PK_{k,0} IND_{k,i,rg,0}}{IT_0 e^{\alpha rs_{k,i,rg,0}} PK_{k,0} KS_{k,i,rg,0}} \quad [162]$$

$$A_{k,i,rg} = \frac{IND_{k,i,rg,0}}{IT_0 e^{\alpha rs_{k,i,rg,0}} KS_{k,i,rg,0}} \quad [163]$$

$$PK_{k,t} IND_{k,i,rg,t} = \frac{A_{k,i,rg} IT_t e^{\alpha rs_{k,i,rg,t}} PK_{k,t} KS_{k,i,rg,t}}{\sum_{kj} \sum_j \sum_{rgj} \left(e^{\alpha rs_{kj,j,rgj,t}} A_{kj,j,rgj,t} PK_{kj,t} KS_{kj,j,rgj,t} \right)} \quad [164]$$

$$IND_{k,i,rg,t} = \frac{A_{k,i,rg} IT_t e^{\alpha rs_{k,i,rg,t}} KS_{k,i,rg,t}}{\sum_{kj} \sum_j \sum_{rgj} \left(e^{\alpha rs_{kj,j,rgj,t}} A_{kj,j,rgj,t} PK_{kj,t} KS_{kj,j,rgj,t} \right)} \quad [165]$$

$$\frac{IND_{k,i,rg,t}}{IT_t} = \frac{A_{k,i,rg} e^{\alpha rs_{k,i,rg,t}} KS_{k,i,rg,t}}{\sum_{kj} \sum_j \sum_{rgj} \left(e^{\alpha rs_{kj,j,rgj,t}} A_{kj,j,rgj,t} PK_{kj,t} KS_{kj,j,rgj,t} \right)} \quad [166]$$

$$\frac{PK_s I_{irs}}{S_r} = \frac{A_{irs} PK_s K_{is} e^{\alpha wk_{is}}}{\sum_{i,s} A_{irs} PK_s K_{is} e^{\alpha wk_{is}}} \quad [167]$$

$$\frac{PK_{k,t} IND_{k,i,rg,t}}{IT_t} = \frac{A_{k,i,rg} PK_{k,t} KS_{k,i,rg,t} e^{\alpha rs_{k,i,rg,t}}}{\sum_{kj} \sum_j \sum_{rgj} A_{kj,j,rgj} PK_{kj,t} KS_{kj,j,rgj,t} e^{\alpha rs_{kj,j,rgj,t}}} \quad [168]$$

$$\frac{IND_{k,i,rg,t}}{IT_t} = \frac{A_{k,i,rg} KS_{k,i,rg,t} e^{\alpha rs_{k,i,rg,t}}}{\sum_{kj} \sum_j \sum_{rgj} A_{kj,j,rgj} PK_{kj,t} KS_{kj,j,rgj,t} e^{\alpha rs_{kj,j,rgj,t}}} \quad [169]$$

$$\frac{PK_{k,t} IND_{k,i,rg,t}}{IT_t} = \frac{A_{k,i,rg} PK_{k,t} KS_{k,i,rg,t}}{\sum_{kj} \sum_j \sum_{rgj} A_{kj,j,rgj} PK_{kj,t} KS_{kj,j,rgj,t}} \quad [170]$$

$$IND_{k,i,rg,t} = BA_{k,i,rg} KS_{k,i,rg,t} e^{\alpha rs_{k,i,rg,t}} \quad [171]$$

$$\sum_k \sum_i \sum_{rg} PK_{k,t} IND_{k,i,rg,t} = IT_t \quad [172]$$

$$R_t = \frac{1}{\alpha} \ln \left[\sum_k \sum_i \sum_{rg} \left(\frac{A_{k,i,rg} PK_{k,t} KS_{k,i,rg,t}}{\sum_{kj} \sum_j \sum_{rgj} A_{kj,j,rgj} PK_{kj,t} KS_{kj,j,rgj,t}} e^{\alpha rs_{k,i,rg,t}} \right) \right] \quad [173]$$

$$e^{\alpha R_t} = \sum_k \sum_i \sum_{rg} \left(\frac{A_{k,i,rg} PK_{k,t} KS_{k,i,rg,t}}{\sum_{kj} \sum_j \sum_{rgj} A_{kj,j,rgj} PK_{kj,t} KS_{kj,j,rgj,t}} e^{\alpha rs_{k,i,rg,t}} \right) \quad [174]$$

$$e^{\alpha R_t} \sum_{kj} \sum_j \sum_{rgj} A_{kj,j,rgj} PK_{kj,t} KS_{kj,j,rgj,t} = \sum_k \sum_i \sum_{rg} A_{k,i,rg} PK_{k,t} KS_{k,i,rg,t} e^{\alpha rs_{k,i,rg,t}} \quad [175]$$

$$\frac{PK_{k,t} IND_{k,i,rg,t}}{IT_t} = \frac{A_{k,i,rg} PK_{k,t} KS_{k,i,rg,t} e^{\alpha rs_{k,i,rg,t}}}{e^{\alpha R_t} \sum_{kj} \sum_j \sum_{rgj} A_{kj,j,rgj} PK_{kj,t} KS_{kj,j,rgj,t}} \quad [176]$$

$$\frac{PK_{k,t} IND_{k,i,rg,t}}{IT_t} = \frac{A_{k,i,rg} PK_{k,t} KS_{k,i,rg,t}}{\sum_{kj} \sum_j \sum_{rgj} A_{kj,j,rgj} PK_{kj,t} KS_{kj,j,rgj,t}} e^{\alpha (rs_{k,i,rg,t} - R_t)} \quad [177]$$

$$B_t = \frac{e^{-\alpha R_t} IT_t}{\sum_{kj} \sum_j \sum_{rgj} A_{kj,j,rgj} PK_{kj,t} KS_{kj,j,rgj,t}} \quad [178]$$

$$v_{k,i,rg,t} = \ln(A_{k,i,rg} PK_{k,t} KS_{k,i,rg,t}) + \alpha rs_{k,i,rg,t} \quad [179]$$

$$\theta_{it} = \frac{\theta_{i0} \left(\frac{PR_{it}}{APR_t} \right)}{\sum_j \theta_{i0} \left(\frac{PR_{jt}}{APR_t} \right)} \quad [180]$$

$$\theta_{i0} = \frac{R_{i0} KD_{i0}}{\sum_j R_{j0} KD_{j0}} \quad [181]$$

$$PR_{it} = \frac{R_{it} KD_{it} - \delta KD_{it} PK_t}{KD_{it} PK_t} \quad [182]$$

$$APR_t = \sum_i \left(\frac{KD_{it} PK_t}{\sum_j KD_{jt} PK_t} PR_{it} \right) \quad [183]$$

$$APR_t = \sum_i \left(\frac{KD_{it}}{\sum_j KD_{jt}} PR_{it} \right) \quad [184]$$

$$KD_{i,t+1} = (1 - \delta)KD_{it} + \theta_{it} IT_t \quad [185]$$

$$\eta_{it} = \left(\frac{KD_{it}}{\sum_j KD_{jt}} \right) \left[\beta_i \left(\frac{R_{it}}{RM_t} - 1 \right) + 1 \right] \quad [186]$$

$$RM_t = \sum_i \left(\frac{KD_{it}}{\sum_j KD_{jt}} R_{it} \right) \quad [187]$$

$$\eta_{it} = \left(\frac{KD_{it}}{\sum_j KD_{jt}} \right) \left[\beta_i \left(\frac{R_{it}}{RM_t} \right) + (1 - \beta_i) \right] \quad [188]$$

$$\eta_{it} = (1 - \beta_i) \left(\frac{KD_{it}}{\sum_j KD_{jt}} \right) + \beta_i \left(\frac{R_{it}}{RM_t} \right) \left(\frac{KD_{it}}{\sum_j KD_{jt}} \right) \quad [189]$$

$$\eta_{it} = \left(\frac{KD_{it}}{\sum_j KD_{jt}} \right) \left(\frac{R_{it}}{RM_t} \right) \quad [190]$$

$$IND_{it} = \eta_{it} \left(\frac{\sum_j PC_{jt} INV_{jt}}{PK_t} \right) \quad [191]$$

$$PK_t = \frac{\sum_j PC_{jt} INV_{jt}}{\sum_j INV_{jt}} \quad [192]$$

$$KD_{i,t+1} = KD_{it} \left(1 + \frac{IND_{it}}{KD_{it}} - \delta \right) \quad [193]$$

$$KD_{i,t+1} = (1 - \delta)KD_{it} + IND_{it} \quad [194]$$

$$H_{i,t+1} = SP_{it} + \mu SP_{it} \left(\frac{R_{it} - AR_t}{AR_t} \right) \quad [195]$$

$$SP_{it} = \frac{R_{it} KD_{it}}{\sum_j R_{jt} KD_{jt}} : \text{industry shares of profits;} \quad [196]$$

$$H_{i,t+1} = SP_{it} \left[\mu \left(\frac{R_{it}}{AR_t} - 1 \right) + 1 \right] \quad [197]$$

$$K_{i,t+1} = (1 - dep) K_{i,t} + \theta_i IT_t \quad [198]$$

$$\frac{ro_i}{uo_i} = 1 \quad [199]$$

$$\frac{INV_{it}}{K_{it}} = A_i \left(\frac{KINC_{it}}{PK_{it} K_{it} (r_t + \delta)} \right)^{\beta_i} = g + \delta \quad [200]$$

$$A_i = (g + \delta) \left(\frac{KINC_{it}}{PK_{it} K_{it} (r_t + \delta)} \right)^{-\beta_i} \quad [201]$$

$$MAX V = \sum_{t=0}^{\infty} \frac{1}{(1+r)^t} [p_t F(K_t, L_t) - w_t L_t - q_t I_t - C(I_t, K_t)] \quad [202]$$

$$MAX V = \sum_{t=0}^{\infty} \frac{1}{(1+r)^t} \left[p_t F(K_t, L_t) - w_t L_t - q_t I_t - q_t \frac{\gamma}{2} \frac{I_t^2}{K_t} \right] \quad [203]$$

$$MAX V = \sum_{t=0}^{\infty} \frac{1}{(1+r)^t} \left[p_t F(K_t, L_t) - w_t L_t - q_t I_t \left(1 + \frac{\gamma}{2} \frac{I_t}{K_t} \right) \right] \quad [204]$$

$$\Lambda = \sum_{t=0}^{\infty} \frac{1}{(1+r)^t} \left\{ p_t F(K_t, L_t) - w_t L_t - q_t I_t \left(1 + \frac{\gamma}{2} \frac{I_t}{K_t} \right) + \lambda_t [I_t - K_{t+1} + (1 - \delta) K_t] \right\} \quad [205]$$

$$- \mu (K_0 - \bar{K}_0)$$

$$\frac{\partial \Lambda}{\partial L_t} = \frac{1}{(1+r)^t} \left[p_t \frac{\partial F}{\partial L_t} - w_t \right] = 0 \quad [206]$$

$$\frac{\partial \Lambda}{\partial I_t} = \frac{1}{(1+r)^t} \left[q_t \left(-1 - \gamma \frac{I_t}{K_t} \right) + \lambda_t \right] = \frac{1}{(1+r)^t} \left[-q_t \left(1 + \gamma \frac{I_t}{K_t} \right) + \lambda_t \right] = 0 \quad [207]$$

$$\frac{\partial \Lambda}{\partial K_t} = \frac{1}{(1+r)^t} \left[p_t \frac{\partial F}{\partial K_t} + q_t \frac{\gamma}{2} \frac{I_t^2}{K_t^2} - (1+r)\lambda_{t-1} + (1-\delta)\lambda_t \right] = 0 \quad [208]$$

$$\frac{\partial \Lambda}{\partial \lambda_t} = \frac{1}{(1+r)^t} [I_t - K_{t+1} + (1-\delta)K_t] \quad [209]$$

$$\frac{\partial \Lambda}{\partial \mu} = -(K_0 - \bar{K}_0) = 0 \quad [210]$$

$$\lambda_t = q_t \left(1 + \gamma \frac{I_t}{K_t} \right) \quad [211]$$

$$\frac{\partial \Lambda}{\partial K_t} = \frac{1}{(1+r)^t} \left[p_t \frac{\partial F}{\partial K_t} + q_t \frac{\gamma}{2} \frac{I_t^2}{K_t^2} - (1+r)q_{t-1} \left(1 + \gamma \frac{I_{t-1}}{K_{t-1}} \right) + (1-\delta)q_t \left(1 + \gamma \frac{I_t}{K_t} \right) \right] = 0 \quad [212]$$

$$p_t \frac{\partial F}{\partial K_t} + q_t \frac{\gamma}{2} \frac{I_t^2}{K_t^2} - (1+r)q_{t-1} \left(1 + \gamma \frac{I_{t-1}}{K_{t-1}} \right) + (1-\delta)q_t \left(1 + \gamma \frac{I_t}{K_t} \right) = 0 \quad [213]$$

$$Q_t = \frac{\partial}{\partial I_t} \left[q_t I_t \left(1 + \gamma \frac{I_t}{K_t} \right) \right] = q_t \left(1 + \gamma \frac{I_t}{K_t} \right) \quad [214]$$

$$p_t \frac{\partial F}{\partial K_t} + q_t \frac{\gamma}{2} \frac{I_t^2}{K_t^2} = (1+r)Q_{t-1} - (1-\delta)Q_t \quad [215]$$

$$(Q_t - Q_{t-1}) = \frac{(Q_t - Q_{t-1})}{Q_{t-1}} Q_{t-1} = \Pi_t Q_{t-1} \quad [216]$$

$$\left(p_t \frac{\partial F}{\partial K_t} + q_t \frac{\gamma}{2} \frac{I_t^2}{K_t^2} \right) = r Q_{t-1} + \delta Q_t - (Q_t - Q_{t-1}) \quad [217]$$

$$\left(p_t \frac{\partial F}{\partial K_t} + q_t \frac{\gamma}{2} \frac{I_t^2}{K_t^2} \right) = (r - \Pi_t) Q_{t-1} + \delta Q_t \quad [218]$$

$$q_t \frac{\gamma}{2} \frac{I_t^2}{K_t^2} = - \frac{\partial}{\partial K_t} \left(q_t \frac{\gamma}{2} \frac{I_t^2}{K_t} \right) = - \frac{\partial C(I_t, K_t)}{\partial K_t} \quad [219]$$

$$\frac{\partial \Phi_t}{\partial K_t} = p_t \frac{\partial F}{\partial K_t} - \frac{\partial C(I_t, K_t)}{\partial K_t} = \left(p_t \frac{\partial F}{\partial K_t} + q_t \frac{\gamma}{2} \frac{I_t^2}{K_t^2} \right) \quad [220]$$

$$\frac{\partial \Phi_t}{\partial K_t} = (r - \Pi_t) Q_{t-1} + \delta Q_t = U_t \quad [221]$$

$$\frac{\partial \Phi_t}{\partial K_t} = (1+r)Q_{t-1} - (1-\delta)Q_t \quad [222]$$

$$(1+r)Q_{t-1} = (1-\delta)Q_t + \frac{\partial \Phi_t}{\partial K_t} \quad [223]$$

$$(1+r)Q_t = (1-\delta)Q_{t+1} + \frac{\partial \Phi_{t+1}}{\partial K_{t+1}} \quad [224]$$

$$Q_t = \frac{1}{(1+r)} \left[(1-\delta)Q_{t+1} + \frac{\partial \Phi_{t+1}}{\partial K_{t+1}} \right] \quad [225]$$

$$Q_t K_{t+1} = \frac{1}{(1+r)} \left[(1-\delta)Q_{t+1} K_{t+1} + \frac{\partial \Phi_{t+1}}{\partial K_{t+1}} K_{t+1} \right] \quad [226]$$

$$Q_t K_{t+1} = \frac{1}{(1+r)} \left(Q_{t+1} (K_{t+2} - I_{t+1}) + \frac{\partial \Phi_{t+1}}{\partial K_{t+1}} K_{t+1} \right) \quad [227]$$

$$Q_t K_{t+1} = \frac{1}{(1+r)} \left(Q_{t+1} K_{t+2} - Q_{t+1} I_{t+1} + \frac{\partial \Phi_{t+1}}{\partial K_{t+1}} K_{t+1} \right) \quad [228]$$

$$Q_t K_{t+1} = \frac{1}{(1+r)} \left(\frac{\partial \Phi_{t+1}}{\partial K_{t+1}} K_{t+1} - Q_{t+1} I_{t+1} + Q_{t+1} K_{t+2} \right) \quad [229]$$

$$Q_t K_{t+1} = \frac{1}{(1+r)} \left[\frac{\partial \Phi_{t+1}}{\partial K_{t+1}} K_{t+1} - Q_{t+1} I_{t+1} + \frac{1}{(1+r)} \left(\frac{\partial \Phi_{t+2}}{\partial K_{t+2}} K_{t+2} - Q_{t+2} I_{t+2} + Q_{t+2} K_{t+3} \right) \right] \quad [230]$$

$$Q_t K_{t+1} = \sum_{s=1}^{\infty} \frac{1}{(1+r)^s} \left[\frac{\partial \Phi_{t+s}}{\partial K_{t+s}} K_{t+s} - Q_{t+s} I_{t+s} \right] + \lim_{s \rightarrow \infty} \left(\frac{1}{(1+r)^s} Q_{t+s} K_{t+s+1} \right) \quad [231]$$

$$Q_t K_{t+1} = \sum_{s=1}^{\infty} \frac{1}{(1+r)^s} \left[\frac{\partial \Phi_{t+s}}{\partial K_{t+s}} K_{t+s} - Q_{t+s} I_{t+s} \right] \quad [232]$$

$$\frac{\partial F}{\partial K_t} K_t = F(K_t, L_t) - \frac{\partial F}{\partial L_t} L_t \quad [233]$$

$$p_t \frac{\partial F}{\partial K_t} K_t = p_t F(K_t, L_t) - p_t \frac{\partial F}{\partial L_t} L_t \quad [234]$$

$$\frac{\partial \Phi_t}{\partial K_t} K_t = \left(p_t F(K_t, L_t) - w_t L_t + q_t \frac{\gamma}{2} \frac{I_t^2}{K_t^2} K_t \right) = \left(p_t F(K_t, L_t) - w_t L_t + q_t \frac{\gamma}{2} \frac{I_t^2}{K_t} \right) \quad [235]$$

$$Q_t K_{t+1} = \sum_{s=1}^{\infty} \frac{1}{(1+r)^s} \left[p_{t+s} F(K_{t+s}, L_{t+s}) - w_{t+s} L_{t+s} + q_{t+s} \frac{\gamma}{2} \frac{I_{t+s}^2}{K_{t+s}} - Q_{t+s} I_{t+s} \right] \quad [236]$$

$$Q_t K_{t+1} = \sum_{s=1}^{\infty} \frac{1}{(1+r)^s} \left[p_{t+s} F(K_{t+s}, L_{t+s}) - w_{t+s} L_{t+s} + q_{t+s} \frac{\gamma}{2} \frac{I_{t+s}^2}{K_{t+s}} - q_{t+s} \left(1 + \gamma \frac{I_{t+s}}{K_{t+s}} \right) I_{t+s} \right] \quad [237]$$

$$Q_t K_{t+1} = \sum_{s=1}^{\infty} \frac{1}{(1+r)^s} \left\{ p_{t+s} F(K_{t+s}, L_{t+s}) - w_{t+s} L_{t+s} + q_{t+s} I_{t+s} \left[\frac{\gamma}{2} \frac{I_{t+s}}{K_{t+s}} - \left(1 + \gamma \frac{I_{t+s}}{K_{t+s}} \right) \right] \right\} \quad [238]$$

$$Q_t K_{t+1} = \sum_{s=1}^{\infty} \frac{1}{(1+r)^s} \left[p_{t+s} F(K_{t+s}, L_{t+s}) - w_{t+s} L_{t+s} - q_{t+s} I_{t+s} \left(1 + \frac{\gamma}{2} \frac{I_{t+s}}{K_{t+s}} \right) \right] \quad [239]$$

$$(1-\delta)Q_t = (1+r)Q_{t-1} - \left(p_t \frac{\partial F}{\partial K_t} + q_t \frac{\gamma}{2} \frac{I_t^2}{K_t^2} \right) \quad [240]$$

$$Q_t = \frac{(1+r)}{(1-\delta)} Q_{t-1} - \frac{1}{(1-\delta)} \left(p_t \frac{\partial F}{\partial K_t} + q_t \frac{\gamma}{2} \frac{I_t^2}{K_t^2} \right) \quad [241]$$

$$Q_{t+1} = \frac{(1+r)}{(1-\delta)} Q_t - \frac{1}{(1-\delta)} \left(p_{t+1} \frac{\partial F}{\partial K_{t+1}} + q_t \frac{\gamma}{2} \frac{I_{t+1}^2}{K_{t+1}^2} \right) \quad [242]$$

$$Q_t = \frac{1}{(1+r)} \left\{ (1-\delta)Q_{t+1} + \left(p_{t+1} \frac{\partial F}{\partial K_{t+1}} + q_{t+1} \frac{\gamma}{2} \frac{I_{t+1}^2}{K_{t+1}^2} \right) \right\} \quad [243]$$

$$Q_{t+1} = \frac{1}{(1+r)} \left\{ (1-\delta)Q_{t+2} + \left(p_{t+2} \frac{\partial F}{\partial K_{t+2}} + q_{t+1} \frac{\gamma}{2} \frac{I_{t+2}^2}{K_{t+2}^2} \right) \right\} \quad [244]$$

$$Q_{t+2} = \frac{1}{(1+r)} \left\{ (1-\delta)Q_{t+3} + \left(p_{t+3} \frac{\partial F}{\partial K_{t+3}} + q_{t+1} \frac{\gamma}{2} \frac{I_{t+3}^2}{K_{t+3}^2} \right) \right\} \quad [245]$$

$$Q_t = \frac{1}{(1+r)} \left\{ \begin{aligned} & \left(\frac{(1-\delta)^3}{(1+r)^2} Q_{t+3} + \frac{(1-\delta)^2}{(1+r)^2} \left(p_{t+3} \frac{\partial F}{\partial K_{t+3}} + q_{t+3} \frac{\gamma}{2} \frac{I_{t+3}^2}{K_{t+3}^2} \right) \right) \\ & + \frac{(1-\delta)}{(1+r)} \left(p_{t+2} \frac{\partial F}{\partial K_{t+2}} + q_{t+2} \frac{\gamma}{2} \frac{I_{t+2}^2}{K_{t+2}^2} \right) \\ & + \left(p_{t+1} \frac{\partial F}{\partial K_{t+1}} + q_{t+1} \frac{\gamma}{2} \frac{I_{t+1}^2}{K_{t+1}^2} \right) \end{aligned} \right\} \quad [246]$$

$$Q_t = \frac{1}{(1+r)} \left\{ \begin{aligned} & \left(\frac{(1-\delta)^s}{(1+r)^{s-1}} Q_{t+s} + \frac{(1-\delta)^{s-1}}{(1+r)^{s-1}} \left(p_{t+s} \frac{\partial F}{\partial K_{t+s}} + q_{t+s} \frac{\gamma}{2} \frac{I_{t+s}^2}{K_{t+s}^2} \right) \right) \\ & \vdots \\ & + \frac{(1-\delta)^2}{(1+r)^2} \left(p_{t+3} \frac{\partial F}{\partial K_{t+3}} + q_{t+3} \frac{\gamma}{2} \frac{I_{t+3}^2}{K_{t+3}^2} \right) \\ & + \frac{(1-\delta)}{(1+r)} \left(p_{t+2} \frac{\partial F}{\partial K_{t+2}} + q_{t+2} \frac{\gamma}{2} \frac{I_{t+2}^2}{K_{t+2}^2} \right) \\ & + \left(p_{t+1} \frac{\partial F}{\partial K_{t+1}} + q_{t+1} \frac{\gamma}{2} \frac{I_{t+1}^2}{K_{t+1}^2} \right) \end{aligned} \right\} \quad [247]$$

$$Q_t = \frac{1}{(1+r)} \sum_{s=1}^{\infty} \left(\frac{1-\delta}{1+r} \right)^{s-1} \left(p_{t+s} \frac{\partial F}{\partial K_{t+s}} + q_{t+s} \frac{\gamma}{2} \frac{I_{t+s}^2}{K_{t+s}^2} \right) \quad [248]$$

$$\frac{\partial C(I_t, K_t)}{\partial K_t} = -q_t \frac{\gamma}{2} \frac{I_t^2}{K_t^2} = -\frac{C(I_t, K_t)}{K_t} \quad [249]$$

$$Q_t = \frac{1}{(1+r)} \sum_{s=1}^{\infty} \left(\frac{1-\delta}{1+r} \right)^{s-1} \left(R_{t+s} - \frac{\partial C(I_{t+s}, K_{t+s})}{\partial K_{t+s}} \right) \quad [250]$$

$$Q_t = \frac{1}{(1+r)} \sum_{s=0}^{\infty} \left(\frac{1-\delta}{1+r} \right)^s \left(R_{t+s+1} - \frac{\partial C(I_{t+s+1}, K_{t+s+1})}{\partial K_{t+s+1}} \right) \quad [251]$$

$$Q_t = \frac{1}{(1+r)} \sum_{s=0}^{\infty} \left(\frac{1-\delta}{1+r} \right)^s \left(R_{t+s+1} + \frac{C(I_{t+s+1}, K_{t+s+1})}{K_{t+s+1}} \right) \quad [252]$$

$$\frac{I_t}{K_t} = \frac{1}{\gamma} \left(\frac{Q_t}{q_t} - 1 \right) \quad [253]$$

$$\frac{C(I_{t+s}, K_{t+s})}{K_{t+s}} = q_{t+s} \frac{\gamma}{2} \frac{I_{t+s}^2}{K_{t+s}^2} \quad [254]$$

$$\frac{C(I_t, K_t)}{K_t} = q_t \frac{\gamma}{2} \frac{I_t^2}{K_t^2} \quad [255]$$

$$Q_t = \frac{1}{(1+r)} \sum_{s=0}^{\infty} \left(\frac{1-\delta}{1+r} \right)^s \left(R_t + q_t \frac{\gamma}{2} \tilde{g}^2 \right) \quad [256]$$

$$Q_t = \frac{1}{(1+r)} \left(\frac{1}{1 - \frac{1-\delta}{1+r}} \right) \left(R_t + q_t \frac{\gamma}{2} \tilde{g}^2 \right) \quad [257]$$

$$Q_t = \frac{1}{(1+r)} \frac{1}{\left(\frac{r+\delta}{1+r} \right)} \left(R_t + q_t \frac{\gamma}{2} \tilde{g}^2 \right) \quad [258]$$

$$Q_t = \frac{1}{(r+\delta)} \left(R_t + q_t \frac{\gamma}{2} \tilde{g}^2 \right) \quad [259]$$

$$\frac{I_t}{K_t} = \frac{1}{\gamma} \left[\frac{1}{q_t(r+\delta)} \left(R_t + q_t \frac{\gamma}{2} \tilde{g}^2 \right) - 1 \right] \quad [260]$$

$$\tilde{g} = \frac{I_t}{K_t} \quad [261]$$

$$\tilde{g} = \frac{1}{\gamma} \left[\frac{1}{q_t(r+\delta)} \left(R_t + q_t \frac{\gamma}{2} \tilde{g}^2 \right) - 1 \right] \quad [262]$$

$$(1 + \gamma \tilde{g}) = \frac{1}{q_t(r+\delta)} \left(R_t + q_t \frac{\gamma}{2} \tilde{g}^2 \right) \quad [263]$$

$$(1 + \gamma \tilde{g}) = \frac{R_t}{q_t(r+\delta)} + \frac{1}{q_t(r+\delta)} q_t \frac{\gamma}{2} \tilde{g}^2 \quad [264]$$

$$(1 + \gamma \tilde{g}) = \frac{R_t}{q_t(r+\delta)} + \frac{1}{(r+\delta)} \frac{\gamma}{2} \tilde{g}^2 \quad [265]$$

$$\frac{1}{(r+\delta)} \frac{\gamma}{2} \tilde{g}^2 - \gamma \tilde{g} + \left(\frac{R_t}{q_t(r+\delta)} - 1 \right) = 0 \quad [266]$$

$$A = \frac{1}{(r+\delta)} \frac{\gamma}{2} \quad [267]$$

$$B = -\gamma \quad [268]$$

$$C = \frac{R_t}{q_t(r+\delta)} - 1 \quad [269]$$

$$\frac{I_t}{K_t} = \tilde{g} = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A} \quad [270]$$

$$\frac{I_t}{K_t} = \tilde{g} = \frac{\gamma \pm \sqrt{\gamma^2 - 4 \frac{1}{(r+\delta)} \frac{\gamma}{2} \left(\frac{R_t}{q_t(r+\delta)} - 1 \right)}}{2 \frac{1}{(r+\delta)} \frac{\gamma}{2}} \quad [271]$$

$$\frac{I_t}{K_t} = \tilde{g} = \frac{1 \pm \sqrt{1 - \frac{2}{\gamma} \frac{1}{(r+\delta)} \left(\frac{R_t}{q_t(r+\delta)} - 1 \right)}}{\frac{1}{(r+\delta)}} \quad [272]$$

$$\frac{I_t}{K_t} = \tilde{g} = (r+\delta) \left[1 \pm \sqrt{1 - \frac{2}{\gamma} \frac{1}{(r+\delta)} \left(\frac{R_t}{q_t(r+\delta)} - 1 \right)} \right] \quad [273]$$

$$\Lambda = \sum_{t=0}^{\infty} \frac{1}{(1+r)^t} [p_t F(K_t, L_t) - w_t L_t - q_t (K_{t+1} - (1-\delta)K_t)] - \lambda (K_0 - \bar{K}_0) \quad [274]$$

$$\frac{\partial \Lambda}{\partial I_t} = \frac{1}{(1+r)^t} (-q_t + \lambda_t) = 0 \quad [275]$$

$$\frac{\partial \Lambda}{\partial K_t} = \frac{1}{(1+r)^t} \left[p_t \frac{\partial F}{\partial K_t} - (1+r)q_{t-1} + q_t(1-\delta) \right] = 0 \quad [276]$$

$$\lambda_t = q_t \quad [277]$$

$$\text{MAX} [p_t F(K_t, L_t) - w_t L_t - u_t K_t] \quad [278]$$

$$p_t \frac{\partial F}{\partial K_t} = u_t \quad [279]$$

$$(1+r)q_{t-1} = (1-\delta)q_t + p_t \frac{\partial F}{\partial K_t} \quad [280]$$

$$(1+r)q_t = (1-\delta)q_{t+1} + p_{t+1} \frac{\partial F}{\partial K_{t+1}} \quad [281]$$

$$q_t = \frac{1}{(1+r)} \left[(1-\delta)q_{t+1} + p_{t+1} \frac{\partial F}{\partial K_{t+1}} \right] \quad [282]$$

$$q_t K_{t+1} = \frac{1}{(1+r)} \left(q_{t+1} (K_{t+2} - I_{t+1}) + p_{t+1} \frac{\partial F}{\partial K_{t+1}} K_{t+1} \right) \quad [283]$$

$$q_t K_{t+1} = \frac{1}{(1+r)} \left(q_{t+1} K_{t+2} - q_{t+1} I_{t+1} + p_{t+1} \frac{\partial F}{\partial K_{t+1}} K_{t+1} \right) \quad [284]$$

$$q_t K_{t+1} = \frac{1}{(1+r)} \left[\begin{aligned} & p_{t+1} \frac{\partial F}{\partial K_{t+1}} K_{t+1} - q_{t+1} I_{t+1} \\ & + \frac{1}{(1+r)} \left(p_{t+2} \frac{\partial F}{\partial K_{t+2}} K_{t+2} - q_{t+2} I_{t+2} + q_{t+2} K_{t+3} \right) \end{aligned} \right] \quad [285]$$

$$q_{t+1} = \frac{1}{(1+r)} \left\{ (1-\delta)q_{t+2} + p_{t+2} \frac{\partial F}{\partial K_{t+2}} \right\} \quad [286]$$

$$q_{t+2} = \frac{1}{(1+r)} \left\{ (1-\delta)q_{t+3} + p_{t+3} \frac{\partial F}{\partial K_{t+3}} \right\} \quad [287]$$

$$q_t = \frac{1}{(1+r)} \left\{ \begin{aligned} & \frac{(1-\delta)^s}{(1+r)^{s-1}} q_{t+s} + \frac{(1-\delta)^{s-1}}{(1+r)^{s-1}} p_{t+s} \frac{\partial F}{\partial K_{t+s}} \\ & \vdots \\ & + \frac{(1-\delta)^2}{(1+r)^2} p_{t+3} \frac{\partial F}{\partial K_{t+3}} \\ & + \frac{(1-\delta)}{(1+r)} p_{t+2} \frac{\partial F}{\partial K_{t+2}} \\ & + p_{t+1} \frac{\partial F}{\partial K_{t+1}} \end{aligned} \right\} \quad [288]$$

$$q_t = \frac{1}{(1+r)} \left\{ \begin{aligned} & \frac{(1-\delta)^\theta}{(1+r)^{\theta-1}} q_{t+\theta} \\ & + \sum_{s=1}^{\theta} \frac{(1-\delta)^{s-1}}{(1+r)^{s-1}} p_{t+s} \frac{\partial F}{\partial K_{t+s}} \end{aligned} \right\} \quad [289]$$

$$q_t = \frac{(1-\delta)^\theta}{(1+r)^\theta} q_\theta + \frac{1}{(1+r)} \sum_{s=1}^{\theta} \frac{(1-\delta)^{s-1}}{(1+r)^{s-1}} p_{t+s} \frac{\partial F}{\partial K_{t+s}} \quad [290]$$

$$q_t = \frac{1}{(r + \delta)} R_t \quad [291]$$

$$\text{MAX } V = \sum_{t=0}^{\infty} \frac{1}{(1+r)^t} \left[p_t F(K_t, L_t) - w_t L_t - q_t l_t - q_t \frac{\gamma}{2} l_t^2 \right] \quad [292]$$

$$\text{MAX } V = \sum_{t=0}^{\infty} \frac{1}{(1+r)^t} \left[p_t F(K_t, L_t) - w_t L_t - q_t l_t \left(1 + \frac{\gamma}{2} l_t \right) \right] \quad [293]$$

$$\Lambda = \sum_{t=0}^{\infty} \frac{1}{(1+r)^t} \left\{ p_t F(K_t, L_t) - w_t L_t - q_t l_t \left(1 + \frac{\gamma}{2} l_t \right) + \lambda_t [l_t - K_{t+1} + (1-\delta)K_t] \right\} - \mu(K_0 - \bar{K}_0) \quad [294]$$

$$\frac{\partial \Lambda}{\partial l_t} = \frac{1}{(1+r)^t} [q_t(-1 - \gamma l_t) + \lambda_t] = \frac{1}{(1+r)^t} [-q_t(1 + \gamma l_t) + \lambda_t] = 0 \quad [295]$$

$$\lambda_{t+1} = q_t(1 + \gamma l_t) \quad [296]$$

$$R_t = (1+r)Q_{t-1} - (1-\delta)Q_t \quad [297]$$

$$(1+r)Q_{t-1} = (1-\delta)Q_t + R_t \quad [298]$$

$$(1+r)Q_t = (1-\delta)Q_{t+1} + R_{t+1} \quad [299]$$

$$Q_t = \frac{1}{(1+r)} [(1-\delta)Q_{t+1} + R_{t+1}] \quad [300]$$

$$Q_t K_{t+1} = \frac{1}{(1+r)} (Q_{t+1} (K_{t+2} - l_{t+1}) + R_{t+1} K_{t+1}) \quad [301]$$

$$Q_t K_{t+1} = \frac{1}{(1+r)} (Q_{t+1} K_{t+2} - Q_{t+1} l_{t+1} + R_{t+1} K_{t+1}) \quad [302]$$

$$Q_t K_{t+1} = \sum_{s=1}^{\infty} \frac{1}{(1+r)^s} [p_{t+s} F(K_{t+s}, L_{t+s}) - w_{t+s} L_{t+s} - q_{t+s} l_{t+s} (1 + \gamma l_{t+s})] \quad [303]$$

$$Q_t K_{t+1} = \sum_{s=1}^{\infty} \frac{1}{(1+r)^s} \left[p_{t+s} F(K_{t+s}, L_{t+s}) - w_{t+s} L_{t+s} - q_{t+s} l_{t+s} \left(1 + \frac{\gamma}{2} l_{t+s} \right) - q_{t+s} l_{t+s} \left(\frac{\gamma}{2} l_{t+s} \right) \right] \quad [304]$$

$$\frac{\sum_{s=1}^{\infty} \frac{1}{(1+r)^s} \left[p_{t+s} F(K_{t+s}, L_{t+s}) - w_{t+s} L_{t+s} - q_{t+s} l_{t+s} \left(1 + \frac{\gamma}{2} l_{t+s} \right) - q_{t+s} l_{t+s} \left(\frac{\gamma}{2} l_{t+s} \right) \right]}{Q_t K_{t+1}} = 1 \quad [305]$$

$$(1-\delta)Q_t = (1+r)Q_{t-1} - p_t \frac{\partial F}{\partial K_t} \quad [306]$$

$$Q_{t+1} = \frac{(1+r)}{(1-\delta)} Q_t - \frac{1}{(1-\delta)} p_{t+1} \frac{\partial F}{\partial K_{t+1}} \quad [307]$$

$$Q_{t+1} = \frac{1}{(1+r)} \left\{ (1-\delta) Q_{t+2} + p_{t+2} \frac{\partial F}{\partial K_{t+2}} \right\} \quad [308]$$

$$Q_{t+2} = \frac{1}{(1+r)} \left\{ (1-\delta) Q_{t+3} + p_{t+3} \frac{\partial F}{\partial K_{t+3}} \right\} \quad [309]$$

$$Q_t = \frac{1}{(1+r)} \left\{ \begin{array}{l} \frac{(1-\delta)^s}{(1+r)^{s-1}} Q_{t+s} + \frac{(1-\delta)^{s-1}}{(1+r)^{s-1}} p_{t+s} \frac{\partial F}{\partial K_{t+s}} \\ \vdots \\ \frac{(1-\delta)^2}{(1+r)^2} p_{t+3} \frac{\partial F}{\partial K_{t+3}} \\ + \frac{(1-\delta)}{(1+r)} p_{t+2} \frac{\partial F}{\partial K_{t+2}} \\ + p_{t+1} \frac{\partial F}{\partial K_{t+1}} \end{array} \right\} \quad [310]$$

$$SH_h = s_h (1+r)^{\beta_{sh}} YD_h \quad [311]$$

$$SM_{men} = SMO_{men} + \psi_{men} YDM_{men} \quad [312]$$

$$PCTL_{l,men,rg} = (1-\psi_{men}) \left[1 - \left(\sum_{gvt} tytemi_{gvt,men}^{TD} + \sum_{gvt} \sum_{prr} tytemi_{gvt,prr}^{TR} \right) \right] (1-TCHO_{l,rg}) w_{l,rg} \quad [313]$$

$$\ln U = \sum_i \gamma_i \ln(C_i - C_i^{MIN}) + \gamma^F \ln\left(\frac{S}{PAF}\right) \quad [314]$$

$$\sum_i P_i C_i + S = YD \quad [315]$$

$$\sum_i \gamma_i + \gamma^F = 1 \quad [316]$$

$$\Lambda = \sum_i \gamma_i \ln(C_i - C_i^{MIN}) + \gamma^F \ln\left(\frac{S}{PAF}\right) - \lambda \left(\sum_i P_i C_i + S - YD \right) \quad [317]$$

$$\frac{\partial \Lambda}{\partial C_i} = \frac{\gamma_i}{(C_i - C_i^{MIN})} - \lambda P_i = 0, \text{ c'est-à-dire } (C_i - C_i^{MIN}) = \frac{\gamma_i}{\lambda P_i} \quad [318]$$

$$\frac{\partial \Lambda}{\partial S} = \frac{\gamma^F}{\left(\frac{S}{PAF}\right)} \frac{1}{PAF} - \lambda = \frac{\gamma^F}{S} - \lambda = 0, \text{ that is, } \frac{\gamma^F}{S} = \lambda \quad [319]$$

$$(C_i - C_i^{MIN}) = \frac{\gamma_i}{\lambda P_i} = \gamma_i \frac{S}{\gamma^F P_i} = S \frac{\gamma_i}{\gamma^F P_i}, \text{ or } C_i = C_i^{MIN} + S \frac{\gamma_i}{\gamma^F P_i} \quad [320]$$

$$\sum_i P_i C_i = \sum_i P_i C_i^{MIN} + \sum_i S \frac{\gamma_i}{\gamma^F} = \sum_i P_i C_i^{MIN} + \frac{1 - \gamma^F}{\gamma^F} S \quad [321]$$

$$YD = \sum_i P_i C_i + S = \sum_i P_i C_i^{MIN} + \frac{1 - \gamma^F}{\gamma^F} S + S = \sum_i P_i C_i^{MIN} + \frac{1}{\gamma^F} S \quad [322]$$

$$CSUP = YD - \sum_i P_i C_i^{MIN} = \frac{1}{\gamma^F} S \quad [323]$$

$$C_i = C_i^{MIN} + \gamma_i \frac{CSUP}{P_i} \quad [324]$$

$$S = \gamma^F CSUP \quad [325]$$

$$CF = \sum_{t=1}^{\infty} \frac{r S}{(1+f)^t PAF} = \frac{r S}{f PAF} \quad [326]$$

$$S = \left(\frac{f}{r} PAF\right) CF \quad [327]$$

$$\ln U = \sum_i \gamma_i \ln(C_i - C_i^{MIN}) + \gamma^F \ln(CF - CF^{MIN}) \quad [328]$$

$$\sum_i P_i C_i + \frac{f PAF}{r} CF = YD \quad [329]$$

$$\Lambda = \sum_i \gamma_i \ln(C_i - C_i^{MIN}) + \gamma^F \ln(CF - CF^{MIN}) - \lambda \left(\sum_i P_i C_i + \frac{f PAF}{r} CF - YD \right) \quad [330]$$

$$\frac{\partial \Lambda}{\partial C_i} = \frac{\gamma_i}{(C_i - C_i^{MIN})} - \lambda P_i = 0, \text{ that is, } (C_i - C_i^{MIN}) = \frac{\gamma_i}{\lambda P_i} \quad [331]$$

$$\frac{\partial \Lambda}{\partial CF} = \frac{\gamma^F}{(CF - CF^{MIN})} - \lambda \frac{f PAF}{r} = 0, \text{ that is, } \lambda = \frac{\gamma^F}{\left(\frac{f PAF}{r}\right)(CF - CF^{MIN})} \quad [332]$$

$$(C_i - C_i^{MIN}) = \frac{\gamma_i}{\lambda P_i} = \gamma_i \frac{\left(\frac{f PAF}{r}\right) (CF - CF^{MIN})}{\gamma^F P_i},$$

$$\text{ou } C_i = C_i^{MIN} + \gamma_i \frac{\left(\frac{f PAF}{r}\right) (CF - CF^{MIN})}{\gamma^F P_i} \quad [333]$$

$$\sum_i P_i C_i = \sum_i P_i C_i^{MIN} + \left(\sum_i \gamma_i\right) \frac{\left(\frac{f PAF}{r}\right) (CF - CF^{MIN})}{\gamma^F}$$

$$= \sum_i P_i C_i^{MIN} + \frac{1 - \gamma^F}{\gamma^F} \left(\frac{f PAF}{r}\right) (CF - CF^{MIN}) \quad [334]$$

$$YD = \sum_i P_i C_i^{MIN} + \left(\frac{f PAF}{r}\right) CF^{MIN} + \frac{1}{\gamma^F} \left(\frac{f PAF}{r}\right) (CF - CF^{MIN}) \quad [335]$$

$$YD = \left[\sum_i P_i C_i^{MIN} + \frac{1 - \gamma^F}{\gamma^F} \left(\frac{f PAF}{r}\right) (CF - CF^{MIN}) \right] + \frac{f PAF}{r} CF \quad [336]$$

$$YD = \left[\sum_i P_i C_i^{MIN} + \frac{1 - \gamma^F}{\gamma^F} \left(\frac{f PAF}{r}\right) (CF - CF^{MIN}) \right] + \frac{f PAF}{r} CF$$

$$- \frac{f PAF}{r} CF^{MIN} + \frac{f PAF}{r} CF^{MIN} \quad [337]$$

$$YD = \left[\sum_i P_i C_i^{MIN} + \frac{1 - \gamma^F}{\gamma^F} \left(\frac{f PAF}{r}\right) (CF - CF^{MIN}) \right]$$

$$+ \frac{f PAF}{r} (CF - CF^{MIN}) + \frac{f PAF}{r} CF^{MIN} \quad [338]$$

$$YD = \sum_i P_i C_i^{MIN} + \left(\frac{f PAF}{r}\right) CF^{MIN} + \left[\frac{1 - \gamma^F}{\gamma^F} \left(\frac{f PAF}{r}\right) + \left(\frac{f PAF}{r}\right) \right] (CF - CF^{MIN}) \quad [339]$$

$$CSUP = YD - \sum_i P_i C_i^{MIN} - \left(\frac{f PAF}{r}\right) CF^{MIN} = \frac{1}{\gamma^F} \left(\frac{f PAF}{r}\right) (CF - CF^{MIN}) \quad [340]$$

$$S = \left(\frac{f PAF}{r}\right) CF = \left(\frac{f PAF}{r}\right) CF^{MIN} + \gamma^F CSUP \quad [341]$$

$$\frac{\partial S}{\partial r} = \frac{\partial S}{\partial \left(\frac{f PAF}{r}\right)} \frac{\partial \left(\frac{f PAF}{r}\right)}{\partial r} = -\frac{1}{r^2} \frac{\partial S}{\partial \left(\frac{f PAF}{r}\right)} \quad [342]$$

$$\frac{\partial S}{\partial r} = -\frac{1}{r^2} \left[CF^{MIN} + \gamma^F \frac{\partial CSUP}{\partial \left(\frac{f PAF}{r} \right)} \right] = -\frac{1}{r^2} (1 - \gamma^F) CF^{MIN} \quad [343]$$

$$\frac{\partial S}{\partial PAF} = \frac{\partial S}{\partial \left(\frac{f PAF}{r} \right)} \frac{\partial \left(\frac{f PAF}{r} \right)}{\partial PAF} = \frac{f}{r} \frac{\partial S}{\partial \left(\frac{f PAF}{r} \right)} = \frac{f}{r} (1 - \gamma^F) CF^{MIN} \quad [344]$$

$$\ln U = \sum_i \gamma_i \ln(C_i - C_i^{MIN}) + \gamma^L \ln(L - L^{MIN}) + \gamma^F \ln(CF - CF^{MIN}) \quad [345]$$

$$\sum_i P_i C_i + wL + PF CF = y + w LS^{MAX} \quad [346]$$

$$\sum_i \gamma_i + \gamma^L + \gamma^F = 1 \quad [347]$$

$$S = PF CF = \left(\frac{f}{r} PAF \right) CF \quad [348]$$

$$\Lambda = \sum_i \gamma_i \ln(C_i - C_i^{MIN}) + \gamma^L \ln(L - L^{MIN}) + \gamma^F \ln(CF - CF^{MIN}) - \lambda \left(\sum_i P_i C_i + wL + PF CF - y - w LS^{MAX} \right) \quad [349]$$

$$CSUPINT = y + w LS^{MAX} - \sum_i P_i C_i^{MIN} - wL^{MIN} - PF CF^{MIN} \quad [350]$$

$$P_i (C_i - C_i^{MIN}) = \gamma_i CSUPINT \quad [351]$$

$$PF (CF - CF^{MIN}) = \gamma^F CSUPINT \quad [352]$$

$$w(L - L^{MIN}) = \gamma^L CSUPINT \quad [353]$$

$$LS = LS^{MAX} - L = LS^{MAX} - L^{MIN} - \gamma^L \frac{CSUPINT}{w} \quad [354]$$

$$MAXHEURES = LS^{MAX} - L^{MIN} \quad [355]$$

$$LS = LS^{MAX} - L = MAXHEURES - \gamma^L \frac{CSUPINT}{w} \quad [356]$$

$$CSUPINT = y + w MAXHEURES - \sum_i P_i C_i^{MIN} - PF CF^{MIN} \quad [357]$$

$$\frac{\partial \Lambda}{\partial C_i} = \frac{\gamma_i}{(C_i - C_i^{MIN})} - \lambda P_i = 0, \text{ that is, } P_i (C_i - C_i^{MIN}) = \frac{\gamma_i}{\lambda} \quad [358]$$

$$\frac{\partial \Lambda}{\partial L} = \left(\frac{\gamma^L}{L - L^{MIN}} \right) - \lambda w = 0, \text{ that is, } w(L - L^{MIN}) = \frac{\gamma^L}{\lambda} \quad [359]$$

$$\frac{\partial \Lambda}{\partial CF} = \left(\frac{\gamma^F}{CF - CF^{MIN}} \right) - \lambda \frac{f PAF}{r} = 0, \text{ that is, } \left(\frac{f PAF}{r} \right) (CF - CF^{MIN}) = \frac{\gamma^F}{\lambda} \quad [360]$$

$$\frac{1}{\lambda} = \frac{1}{\gamma^L} w(L - L^{MIN}) \quad [361]$$

$$P_i(C_i - C_i^{MIN}) = \frac{\gamma_i}{\gamma^L} w(L - L^{MIN}) \quad [362]$$

$$\left(\frac{f PAF}{r} \right) CF = \left(\frac{f PAF}{r} \right) CF^{MIN} + \frac{\gamma^F}{\gamma^L} w(L - L^{MIN}) \quad [363]$$

$$\sum_i P_i C_i = \sum_i P_i C_i^{MIN} + \frac{1}{\gamma^L} w(L - L^{MIN}) \sum_i \gamma_i = \frac{(1 - \gamma^L - \gamma^F)}{\gamma^L} w(L - L^{MIN}) \quad [364]$$

$$y + w LS^{MAX} = \sum_i P_i C_i^{MIN} + \frac{(1 - \gamma^L - \gamma^F)}{\gamma^L} w(L - L^{MIN}) + wL \\ + \left(\frac{f PAF}{r} \right) CF^{MIN} + \frac{\gamma^F}{\gamma^L} w(L - L^{MIN}) \quad [365]$$

$$y + w LS^{MAX} - \sum_i P_i C_i^{MIN} - \left(\frac{f PAF}{r} \right) CF^{MIN} \\ = \frac{(1 - \gamma^L - \gamma^F)}{\gamma^L} w(L - L^{MIN}) + wL + \frac{\gamma^F}{\gamma^L} w(L - L^{MIN}) \quad [366]$$

$$y + w LS^{MAX} - \sum_i P_i C_i^{MIN} - \left(\frac{f PAF}{r} \right) CF^{MIN} - wL^{MIN} \\ = \frac{(1 - \gamma^L - \gamma^F)}{\gamma^L} w(L - L^{MIN}) + w(L - L^{MIN}) + \frac{\gamma^F}{\gamma^L} w(L - L^{MIN}) \quad [367]$$

$$y + w LS^{MAX} - \sum_i P_i C_i^{MIN} - \left(\frac{f PAF}{r} \right) CF^{MIN} - wL^{MIN} = \frac{1}{\gamma^L} w(L - L^{MIN}) \quad [368]$$

$$CSUPINT = \frac{1}{\gamma^L} w(L - L^{MIN}) \quad [369]$$

$$\frac{g_a}{1-g_a} = \psi \left(\frac{1+i_a}{1+i_b} \right)^{\varepsilon_a}, \text{ that is, } g_a = \frac{\psi \left(\frac{1+i_a}{1+i_b} \right)^{\varepsilon_a}}{1 + \psi \left(\frac{1+i_a}{1+i_b} \right)^{\varepsilon_a}} \quad [370]$$

$$U = \left[\sum_i A_i (z_i V_i)^\rho \right]^{\frac{1}{\rho}} \quad [371]$$

$$\text{s.c. } \sum_i V_i = W \quad [372]$$

$$z_i V_i = W \frac{A_i^\sigma z_i^{\sigma-1}}{\zeta}, \text{ where } \zeta = \left[\sum_j A_j^\sigma z_j^{\sigma-1} \right] \quad [373]$$

$$M_\rho = \left(\frac{\sum_i w_i (x_i)^\rho}{\sum_i w_i} \right)^{1/\rho} \quad [374]$$

$$\text{MAX}_{a_i} VC = \sum_i \xi_i a_i, \text{ where } \xi_i = (1+r_i)q_i \quad [375]$$

$$W = A_w \left[\sum_i \delta_i a_i^\beta \right]^{\frac{1}{\beta}} \quad [376]$$

$$\tau = \frac{1}{1-\beta} \quad (\beta > 1) \quad [377]$$

$$\sum_i q_i a_i = W \quad [378]$$

$$q_i a_i = W \frac{\delta_i^\tau q_i \xi_i^{-\tau}}{\sum_j \delta_j^\tau q_j \xi_j^{-\tau}} \quad [379]$$